ELEMENTS OF DESCRIPTIVE GEOMETRY

RANDALL

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ELEMENTS OF DESCRIPTIVE GEOMETRY

WITH APPLICATIONS TO

ISOMETRIC PROJECTION AND OTHER FORMS OF ONE-PLANE PROJECTION

A TEXT-BOOK FOR COLLEGES AND ENGINEERING SCHOOLS

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PREFACE

The aim of this treatise is to make a clear presentation of the theory of projection, to show the application of this theory as a medium of expression, and by the discussion and proof of a great variety of problems to enable the student to make a ready and intelligent use of this medium in the representation of all forms of magnitudes.

As by far the greater part of practical drafting is done from the standpoint of the third quadrant, there seems to be no good reason why the principles of descriptive geometry, which are so directly and extensively applied in practice, should not also be presented from the standpoint of the same quadrant.

Therefore, while the student is called upon to work freely in all the four quadrants, the subject-matter is presented primarily from the third quadrant.

In the establishment of principles great effort is made to be explicit; but in the application of these principles, for which purpose a great many unsolved problems are assigned, the student is left largely to his own resources.

As the principles of projection are fundamental in all branches of drafting, it follows that no attempt at extensive application of these principles in such subjects as machine drawing, gearing, architectural drawing, etc., should be made until the principles themselves have been thoroughly established. For this reason the attention of this work is largely confined to theoretical considerations, although a number of simple practical applications such as the student can safely and intelligently make are introduced.

Free use is made of profile and other supplementary planes of projection.

Isometric projection and other forms of one-plane projection are treated as applications of descriptive geometry.

It is hoped that the system of notation which is introduced will be found both simple and expressive; and that the method of locating given parts which may be employed in the assignment of work in the recitation room and in the drafting room will be found useful.

CONTENTS

DESCRIPTIVE GEOMETRY

CHAPTER I

DEFINITIONS AND ASSUMPTIONS

1. The Subject defined. Descriptive geometry is that branch of mathematics which seeks, through the medium of an exact process of graphic expression, to represent geometrical magnitudes which occupy given positions in space, and also through the same medium of expression to solve such problems as relate to these magnitudes.

2. Representation of Magnitudes of Two Dimensions. A magnitude of two dimensions, such as a plane geometrical figure, may be easily and directly represented, graphically, upon a single plane, since every characteristic of such a magnitude may be determined from a single standpoint of observation, and the whole may be outlined upon the very plane in which the magnitude exists.

The diagrams connected with the statement and solution of problems in plane geometry furnish an illustration of this fact.

3. Representation of Magnitudes of Three Dimensions. A magnitude of three dimensions does not exist in a single plane, neither can its characteristics be completely determined from a single standpoint of observation; therefore the process of representation must necessarily be different from that employed in connection with magnitudes of two dimensions.

4. Projection. Since the points and lines of magnitudes of three dimensions do not exist in a single plane, as is the case with magnitudes of two dimensions, it will be necessary to determine some plane of representation and to establish some process by which reference to this plane may be made.

The plane of representation, or the plane upon which the representation is made, is called the *plane of projection*,* and the process by which reference to this plane is made is called *projection*.

Projection involves the assumption of a point of sight, or position for the observer, and a plane of projection.

When the point of sight and the plane of projection are given, the projection of an object, for simplicity a point, is accomplished when the point is literally thrown forward along its visual ray† until it rests upon the plane. In other words, the projection of a point upon any plane is the intersection of the visual ray of the point with that plane.

The point may occupy a position between the observer and the plane of projection, or the plane of projection may stand between the observer and the point; but in either case the projection of the point is found by the process stated above.

If the point is in the plane of projection, it is evident that the point and its projection will be identical.

5. Systems of Projection. It is evident that the character of the projection of a magnitude, which consists of a collection of points, will depend upon the relative positions of the magnitude, the observer, and the plane of projection.

There are two principal systems of projection, depending upon the position of the observer with reference to the plane of projection, — the *scenographic projection* and the *orthographic projection.*

The scenographic projection is that system in which the point of sight is assumed within a finite distance of the plane of projection.

As this is the position which would naturally be assumed by an observer, the projection upon the plane will correspond with that made upon the retina of the eye, and the picture will be true to nature. This system is employed whenever it is desired to represent an object as it appears, rather than to show its exact dimensions; but on account of difficulties attending the operation of the

* Projection may be made upon various surfaces, such as cylindrical surfaces, spherical surfaces, etc.; but in this work attention will be confined to projection upon plane surfaces.

† A visual ray is any straight line passing through the point of sight.

system, owing to the obliquity of the visual rays to the plane of projection, the system is not practicable for problematic work.

The orthographic projection — the system which will be employed throughout this work — is that system in which the point of sight is assumed at an infinite distance from the plane of projection.

In this system, since magnitudes are assumed within a finite distance of the plane of projection, visual rays to the various points of such magnitudes may be regarded as parallel lines, and may be assumed perpendicular to the plane of projection.

Under these conditions the orthographic projection of a point upon any plane is the point in which a straight line drawn through the point perpendicular to the plane pierces the plane.

6. Planes of Projection. As a rule it is not possible to learn all the characteristics of magnitudes of three dimensions from a single standpoint of observation, and for this reason more than one plane of projection is usually needed.

There are two principal planes of projection, — a horizontal plane called the *horizontal plane of projection* or H, and a vertical plane perpendicular to H and called the *vertical plane of projection* or V.

7. The Ground Line. The intersection of H and V is called the *ground line*, or G–L.

8. Quadrants. The planes H and V divide space into four right dihedral angles known as the *first quadrant*, the *second quadrant*, the *third quadrant*, and the *fourth quadrant*.

The first quadrant is above H and in front of V, the second quadrant is above H and back of V, the third quadrant is below H and back of V, and the fourth quadrant is below H and in front of V.

9. Projecting Lines. In orthographic projection the visual rays are called *projecting lines*, and are assumed perpendicular to the plane on which projection is made.

10. Position of the Observer. When projecting on H the observer is supposed to be above H and at an infinite distance from it.

When projecting on V the observer is supposed to be in front of V and at an infinite distance from it.

11. Horizontal and Vertical Projections. Projections on H are called *horizontal projections*, and projections on V are called *vertical projections*.

12. Revolution of the Planes of Projection. The primary position of H is horizontal, and the primary position of V is vertical, giving horizontal and vertical projections on two distinct planes perpendicular to each other.

In order that projections upon H and V may be represented upon a single plane, one of the planes of projection is revolved about $G–L$ as an axis until it is coincident with the other.

This last position of H and V is called the secondary position, or position of coincidence.

CHAPTER II

REPRESENTATION OF THE POINT, LINE, AND PLANE

13. Projection of the Point upon H and V in their Primary Position. In Fig. 1 let H and V represent the horizontal and vertical planes of projection in their primary position, and let M represent a point in the first quadrant. Then, according to Section 5, the horizontal projection of M, or the projection of M upon H, is $m_{\prime\prime}$; and the vertical projection of M, or the projection of M upon V, is m''. The line M–$m_{\prime\prime}$ is perpendicular to H and is called the horizontal projecting line of M. The line M–m'' is perpendicular to V and is called the vertical projecting line of M.

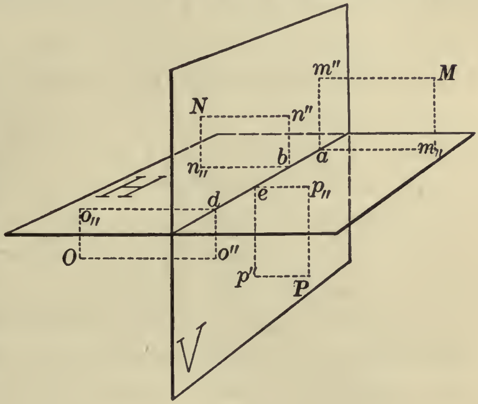

Fig. 1

N is a point in the second quadrant. Its horizontal projection is $n_{\prime\prime}$ and its vertical projection is n''. The line N–$n_{\prime\prime}$ is perpendicular to H and is called the horizontal projecting line of N. The line N–n'' is perpendicular to V and is called the vertical projecting line of N.

O is a point in the third quadrant. Its horizontal projection is $o_{\prime\prime}$, and its vertical projection is o''. The line O–$o_{\prime\prime}$ is perpendicular to H and is called the horizontal projecting line of O. The line O–o'' is perpendicular to V and is called the vertical projecting line of O.

P is a point in the fourth quadrant. Its horizontal projection is $p_{\prime\prime}$ and its vertical projection is p''. The line P–$p_{\prime\prime}$ is perpendicular to H and is called the horizontal projecting line of P. The

line P–p'' is perpendicular to V and is called the vertical projecting line of P.

It will be noticed in Fig. 1 that the horizontal and vertical projecting lines of a point determine a plane which is perpendicular to both H and V and is therefore perpendicular to G–L; also that the straight lines in which this plane intersects H and V are perpendicular to G–L at the same point and pass respectively through the horizontal and vertical projections of the point.

Observe that when the horizontal and vertical projections of a point are given, the point itself is definitely located, for the horizontal and vertical projecting lines, determined by the projections of the point, lie in the same plane and intersect at the only point which can have its horizontal and vertical projections at the points given.

Observe that the distance of a point from H is in each case indicated by the distance of its vertical projection from G–L, and that the distance of the point from V is in each case indicated by the distance of its horizontal projection from G–L.

If a point is situated in H, its horizontal projection is the point itself and its vertical projection is in G–L.

If a point is situated in V, its vertical projection is the point itself and its horizontal projection is in G–L.

If a point is situated in G–L, both its horizontal and vertical projections coincide with the point itself.

14. Representation of the Point upon H and V in their Position of Coincidence. In Fig. 2 the projections $m_{\prime\prime}$, m''; $n_{\prime\prime}$, n''; $o_{\prime\prime}$, o''; and $p_{\prime\prime}$, p'' are those previously found in Fig. 1 and represent points in the first, second, third, and fourth quadrants respectively.

If the plane H be revolved about G–L as an axis until that portion of H back of V falls on V above H, and that portion of H in front of V falls on V below H, the point m'' will remain stationary, while the point $m_{\prime\prime}$ will move in the arc of a circle with a as a center, and fall at $m_{\prime}$ in the line m''–a produced.

The point n'' will remain stationary, while the point $n_{\prime\prime}$ will move in the arc of a circle with b as a center, and fall at $n_{\prime}$ in the line b–n'' produced. The point o'' will remain stationary, while the point $o_{\prime\prime}$ will move in the arc of a circle with d as a center, and fall at $o_{\prime}$ in the line o''–d produced.

The point p'' will remain stationary, while the point $p_{,,}$ will move in the arc of a circle with e as a center, and fall at $p_{,}$ in the line e–p''.

It will be noticed that in the revolution vertical projections remain stationary, while horizontal projections move in arcs of circles with centers in G–L, and fall upon V in straight lines drawn through the corresponding vertical projections perpendicular to G–L.

After the revolution of the plane H, or when the planes are in their position of coincidence, the projections will appear as shown in Fig. 3, where that portion of the plane of the paper above G–L, regarding the plane of the paper as vertical, represents both that portion of V above H and that portion of H back of V; and where that portion of the plane of the paper below G–L represents both that portion of V below H and that portion of H in front of V.

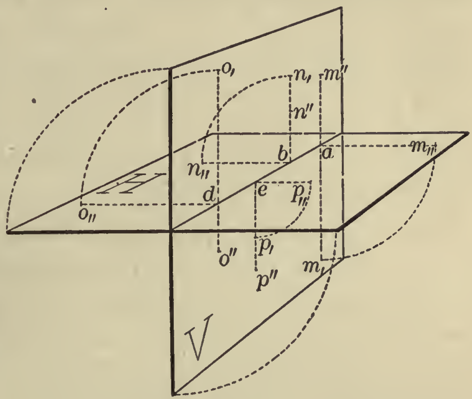

Fig. 2

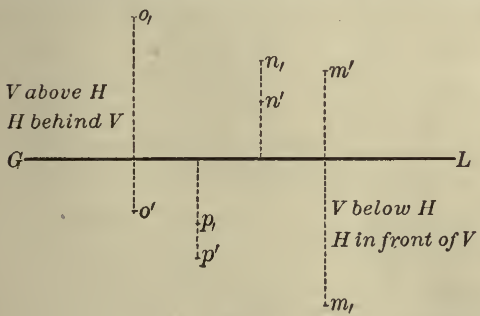

Fig. 3

Again, referring to Fig. 2, if the plane V be revolved about G–L as an axis until that portion of V above H falls upon that portion of H back of V, and that portion of V below H falls upon that portion of H in front of V, the horizontal projections will remain stationary, while the vertical projections will move in arcs of circles with centers in G–L, and fall upon H in straight lines drawn through the corresponding horizontal projections perpendicular to G–L.

After the revolution of the plane V into coincidence with H the projections will again be expressed as shown in Fig. 3, where that portion of the plane of the paper back of G–L, regarding the plane of the paper as horizontal, represents both that portion of H behind V and that portion of V above H, and where that portion of the plane of the paper in front of G–L represents both that portion of H in front of V and that portion of V below H.

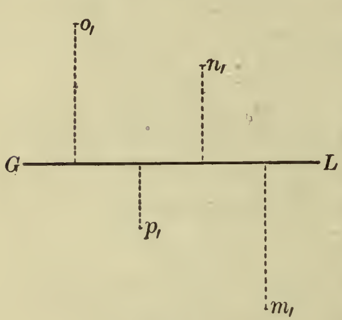

Fig. 4

It will be observed that whether we revolve H into coincidence with V or whether we revolve V into coincidence with H the result will be the same. When working on the blackboard, where the surface is usually vertical, the former method will be found convenient. When working on the drawing board, where the surface is usually horizontal, the latter method will be found more natural.

Again referring to Fig. 1, let us first project the four points M, N, O, and P upon H alone. As the plane V is perpendicular to H it will in this projection appear as a straight line coincident with G–L, and the four projections will appear as shown in Fig. 4, where that portion of the plane of the paper back of G–L represents H back of V, and where that portion of the plane of the paper in front of G–L represents H in front of V.

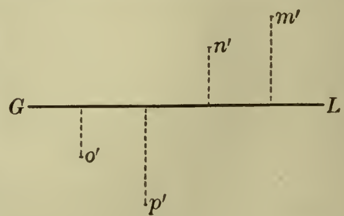

Fig. 5

Now project the four points upon V alone. The plane H in this projection will appear as a straight line coincident with G–L, and the four projections will appear as shown in Fig. 5, where that portion of the plane of the paper above G–L represents V above H, and where that portion of the plane of the paper below G–L represents V below H.

Now if we place Fig. 5 upon Fig. 4 in such a way that the ground line of one shall coincide with the ground line of the

other, and so that the two projections of each point shall fall upon the same straight line perpendicular to $G–L$, the result will be precisely the same as that expressed in Fig. 3.

It will be observed that whether we revolve H into coincidence with V, or V into coincidence with H, or whether the projections are made upon H and V independently and afterwards combined as shown above, the result is the same.

The object of the transformation is to make it possible to represent upon a single plane, projections which belong primarily upon two planes perpendicular to each other.

It is immaterial by which method the transformation is made; the essential thing is that the student shall be able to pass in imagination, without any difficulty or hesitation, from the position of perpendicularity to that of coincidence, and *vice versa.*

Returning now to Fig. 3, which expresses the common result of the three methods of transformation, it will be noticed that when a point, as M, is in the first quadrant, its horizontal projection will be in front of $G–L$ and its vertical projection will be above $G–L$; that when a point, as N, is in the second quadrant, its horizontal projection will be back of $G–L$ and its vertical projection will be above $G–L$; that when a point, as O, is in the third quadrant, its horizontal projection will be back of $G–L$ and its vertical projection will be below $G–L$; that when a point, as P, is in the fourth quadrant, its horizontal projection will be in front of $G–L$ and its vertical projection will be below $G–L$.

Conversely, referring to Fig. 3, if a horizontal projection, as $m_{,}$, is situated in front of $G–L$, it will be known that the point M must be in front of V, that is, either in the first quadrant or in the fourth quadrant; and if the vertical projection, as m', of the same point M is situated above $G–L$, it will be known that the point M must be above H, that is, either in the first quadrant or in the second quadrant, and therefore it will be known that the point M must be in the first quadrant.

If a horizontal projection, as $n_{,}$, is situated back of $G–L$, it will be known that the point N must be back of V, that is, either in the second quadrant or in the third quadrant; and if the corresponding vertical projection, as n', is situated above $G–L$, it will be

known that the point N must be above H, that is, either in the first quadrant or in the second quadrant, and therefore it will be known that the point N must be in the second quadrant.

If a horizontal projection, as $o_{\prime}$, is situated back of $G–L$, it will be known that the point O must be back of V, that is, either in the second quadrant or in the third quadrant; and if the corresponding vertical projection, as o', is situated below $G–L$, it will be known that the point O must be below H, that is, either in the third quadrant or in the fourth quadrant, and therefore it will be known that the point O must be in the third quadrant.

If a horizontal projection, as $p_{\prime}$, is situated in front of $G–L$, it will be known that the point P must be in front of V, that is, either in the first quadrant or in the fourth quadrant; and if the corresponding vertical projection, as p', is situated below $G–L$, it will be known that the point P must be below H, that is, either in the third quadrant or in the fourth quadrant, and therefore it will be known that the point P must be in the fourth quadrant.

It will be observed, from a comparison of Figs. 1, 2, and 3, that the horizontal and vertical projections of a point, in the transformed position shown in Fig. 3, must lie in the same straight line perpendicular to $G–L$.

It will be also observed that in the transformed position the distance of the horizontal projection of a point from $G–L$ still indicates the distance of the point itself from V, and that the distance of the vertical projection of a point from $G–L$ still indicates the distance of the point itself from H.

15. To assume a Point at Random. To assume a point at random, assume its projections at random, provided they fall in the same straight line perpendicular to $G–L$.

16. Problem 1. *Determine the projections of a point situated in the first quadrant,* 2 *units from H and* 4 *units from V.*

17. Problem 2. *Determine the projections of a point in the third quadrant,* 1 *unit from H and* 4 *units from V.*

18. Problem 3. *Determine the projections of a point situated in H and* 3 *units back of V.*

19. Problem 4. *Determine the projections of a point situated in V and* 5 *units below H.*

20. Problem 5. *Determine the projections of a point situated in the fourth quadrant,* 4 *units from H and* 4 *units from V.*

21. Problem 6. — *Determine the projections of a point situated in the second quadrant,* 4 *units from H and* 2 *units from V.*

22. Problem 7. *Determine the projections of a point situated in G–L.*

23. Projection of the Straight Line upon *H* and *V* in their Primary Position. As a straight line may be regarded as made up of an infinite number of consecutive points, the projection of the straight line will be the locus of the projections of these points.

The projecting lines of these points, since they are drawn from points in a straight line and perpendicular to a plane, constitute a plane perpendicular to the plane of projection.

The plane made up of these projecting lines is called the projecting plane of the line. If the projection is being made upon *H*, the projecting plane is called the *horizontal projecting plane;* and if the projection is being made upon *V*, the projecting plane is called the *vertical projecting plane.*

The projection of a straight line upon a plane, then, is a straight line, and for this reason its position is fully known when two of its points are determined.

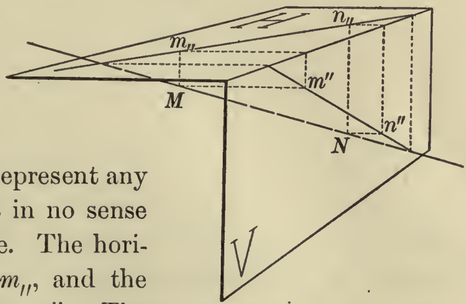

Fig. 6

In Fig. 6, which shows *H* and *V* in their primary position, *M–N* represents a straight line in the third quadrant. *M* and *N* represent any two points of the line but in no sense limit the length of the line. The horizontal projection of *M* is $m_{//}$, and the vertical projection of *M* is m''. The horizontal projection of *N* is $n_{//}$, and the vertical projection of *N* is n''. The horizontal projection of the line is then $m_{//}$–$n_{//}$, and the vertical projection of the line is m''–n''.

The projection of a straight line, then, may be found by finding the projections of two of its points and drawing a straight line through these two projections. This will be true whatever the quadrant occupied by the line.

It will be noticed that when the horizontal and vertical projections of a straight line are given, the line is in general definitely determined, for the horizontal and vertical projecting planes determined by the projections of the line intersect in the only line which can have its horizontal and vertical projections in these lines.

It will be also noticed that the two projections of a line determine the position of the line with reference to H and V. For example, in Fig. 6 the vertical projection of the line shows that the line slopes downward to the right, the horizontal projection shows that the line approaches V as it extends toward the right,

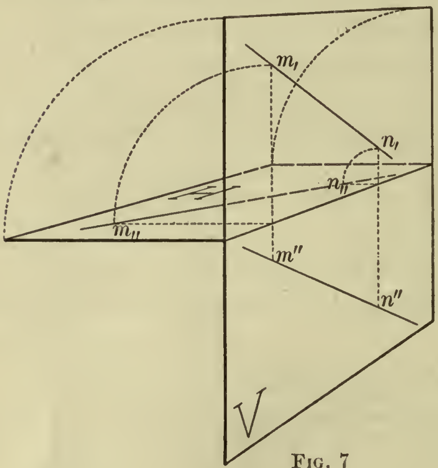

Fig. 7

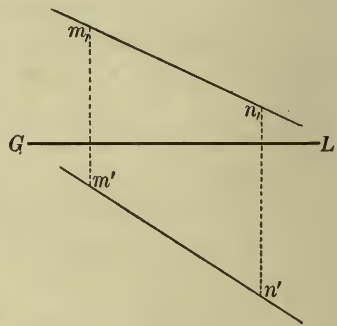

Fig. 8

and the two projections together show that the line slopes downward toward V to the right.

24. Representation of the Straight Line upon *H* and *V* in their Position of Coincidence. In Fig. 7 the projections $m_{\prime\prime}$–$n_{\prime\prime}$ and m''–n'' are those previously found in Fig. 6, and represent a straight line in the third quadrant.

If the plane V remains stationary and the plane H be revolved as previously directed, the points m'' and n'' will remain stationary, while the points $m_{\prime\prime}$ and $n_{\prime\prime}$ will revolve and fall at the points $m_{\prime}$ and $n_{\prime}$ respectively. The line $m_{\prime}$–$n_{\prime}$, then, will represent the horizontal projection of the line M–N after H has been revolved, and the result may be expressed as shown in Fig. 8.

It will be easily seen that we shall obtain the same result if we allow the horizontal plane of projection to remain stationary and revolve V as directed in Section 14.

It will be observed that the two projections of the line in Fig. 8 bear the same relation to G–L as they did in Fig. 6, and therefore reveal just as much in regard to the position of the line itself.

25. Projections of Straight Lines occupying Various Positions with Reference to H and V. In Fig. 9, M–N represents a straight line in the third quadrant perpendicular to H. The horizontal projection is a point $m_{,}$, $n_{,}$. The vertical projection is a straight line m'–n' perpendicular to G–L and parallel to M–N. That portion

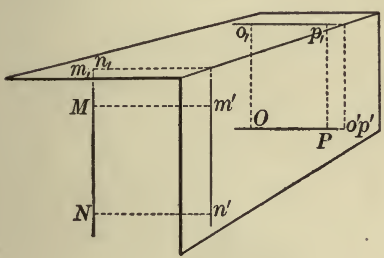

FIG. 9

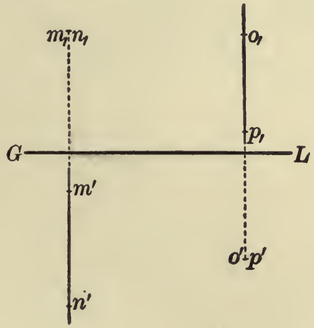

FIG. 10

of the vertical projection included between m' and n' is equal to the distance between M and N.

In the same figure O–P represents a straight line in the third quadrant perpendicular to V. The vertical projection is a point o', p'. The horizontal projection is a straight line $o_{,}$–$p_{,}$ perpendicular to G–L and parallel to O–P. That portion of the horizontal projection included between $o_{,}$ and $p_{,}$ is equal to the distance between O and P.

Fig. 10 shows how the projections of Fig. 9 will appear after H and V take their position of coincidence.

In Fig. 11, M–N represents a straight line in the third quadrant, parallel to H but oblique to V. The vertical projection m'–n' is parallel to G–L and at the same distance below G–L that M–N is

below *H*. The horizontal projection $m_{\prime}$–$n_{\prime}$ is parallel to *M–N*, and makes the same angle with *G–L* that *M–N* makes with *V*. That portion of the horizontal projection included between $m_{\prime}$ and $n_{\prime}$ is equal to the distance between *M* and *N*.

In the same figure *O–P* represents a straight line in the third quadrant, parallel to *V* but oblique to *H*. The horizontal projection $o_{\prime}$–$p_{\prime}$ is parallel to *G–L* and at the same distance back of *G–L* that *O–P* is back of *V*. The vertical projection o'–p' is parallel

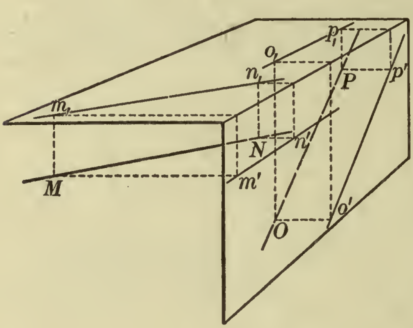

Fig. 11

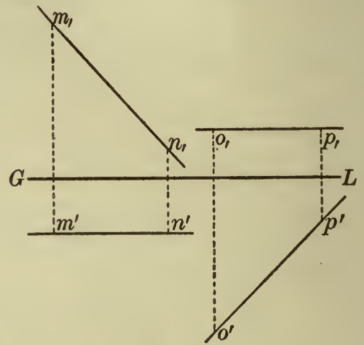

Fig. 12

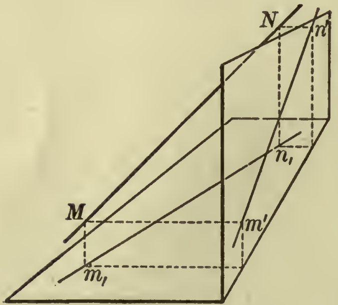

Fig. 13

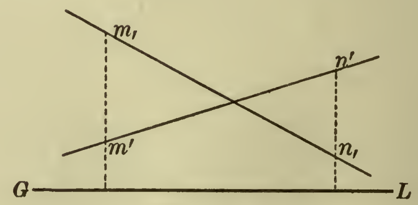

Fig. 14

to *O–P* and makes the same angle with *G–L* that *O–P* makes with *H*. That portion of the vertical projection included between o' and p' is equal to the distance between *O* and *P*.

Fig. 12 shows how the projections of Fig. 11 will appear after *H* and *V* take their position of coincidence.

In Fig. 13, *M–N* represents a straight line in the second quadrant and oblique to both *H* and *V*.

Fig. 14 shows how the projections of Fig. 13 will appear after H and V take their position of coincidence.

In Fig. 15, M–N represents a straight line located by the point M in the third quadrant and by the point N in the first quadrant. The line pierces V at a' and pierces H at $b_{\prime}$. The section M–$b_{\prime}$ of the line is in the third quadrant, the section $b_{\prime}$–a' is in the second quadrant, and the section a'–N is in the first quadrant.

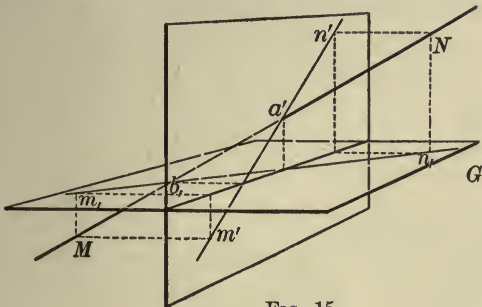

FIG. 15

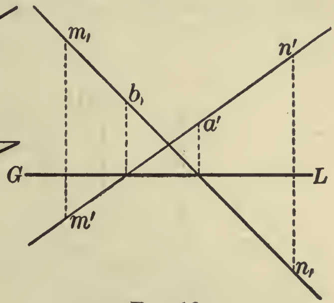

FIG. 16

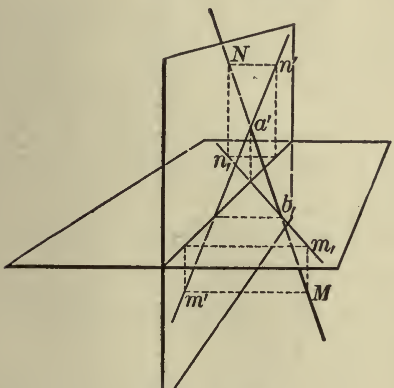

FIG. 17

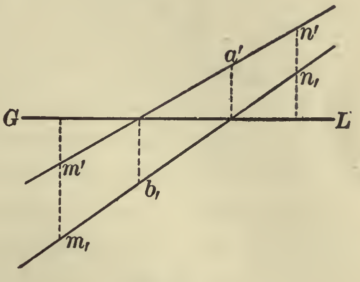

FIG. 18

Fig. 16 shows how the projections of Fig. 15 will appear after H and V take their position of coincidence.

In Fig. 17, M–N represents a straight line located by the point M in the fourth quadrant and by the point N in the second quadrant. The line pierces H at $b_{\prime}$, runs through the first quadrant, and pierces V at a'.

Fig. 18 shows how the projections of Fig. 17 will appear after H and V take their position of coincidence.

In Fig. 19, *M–N* represents a straight line in a profile plane and located by a point *M* in the first quadrant and by a point *N* in the third quadrant.

Fig. 20 shows how the projections of Fig. 19 will appear after *H* and *V* take their position of coincidence.

26. Observations. When a straight line is parallel to a plane of projection, the projection upon this plane of any definite portion of the line will be equal in length to the portion in question.

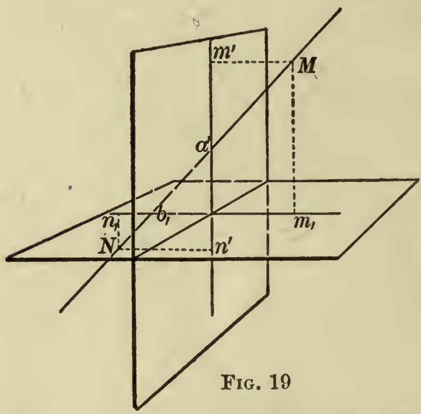

FIG. 19

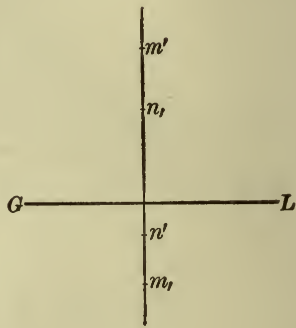

FIG. 20

If a straight line is situated in a plane of projection, the line itself is its own projection upon this plane.

If a straight line is situated in *H*, its horizontal projection is the line itself and its vertical projection is in *G–L*.

If a straight line is situated in *V*, its vertical projection is the line itself and its horizontal projection is in *G–L*.

If a straight line is parallel to *H* and oblique to *V*, its vertical projection will be parallel to *G–L* and its horizontal projection will make the same angle with *G–L* that the line itself makes with *V*.

If a straight line is parallel to *V* and oblique to *H*, its horizontal projection will be parallel to *G–L* and its vertical projection will make the same angle with *G–L* that the line itself makes with *H*.

If a straight line is in a profile plane, its projecting planes coincide and its projections are perpendicular to *G–L*.

27. To assume a Straight Line at Random. To assume a straight line at random, we may, within certain limitations, draw its two projections at random.

28. To assume a Point upon a Straight Line. In Fig. 21 let $m_{,}-n_{,}$ and $m'-n'$ represent the horizontal and vertical projections of a straight line in the third quadrant. Assume one of the projections of the point, say the horizontal, $o_{,}$, anywhere on the line $m_{,}-n_{,}$. Through $o_{,}$ draw a straight line perpendicular to $G-L$ and note its intersection o' with $m'-n'$. The point o' is the vertical projection of the required point.

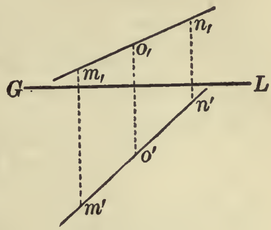

FIG. 21

If a point is situated on a line, its horizontal projection must be on the horizontal projection of the line, and its vertical projection must be on the vertical projection of the line.

29. To assume Two Straight Lines which intersect. See Fig. 22. Draw at random the two projections, $m_{,}-n_{,}$ and $m'-n'$, of one of the lines. Assume upon this line any point as P. Through $p_{,}$ and p' respectively draw at random the horizontal and vertical projections of the other line.

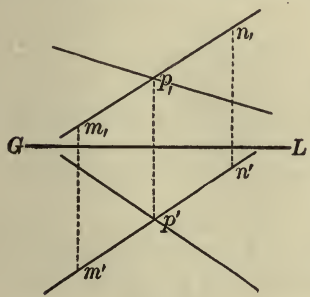

FIG. 22

30. The Projections of Parallel Straight Lines. The projecting planes of parallel straight lines are necessarily parallel whatever the plane of projection; therefore the intersections of these planes with the plane of projection, or in other words the projections of the lines, must be parallel.

31. The Projections of Straight Lines which are Perpendicular to Each Other. The projections of two straight lines which are perpendicular to each other are perpendicular to each other only when one or both of the lines are parallel to the plane of projection, since it is only under these conditions that the projecting planes of the lines are perpendicular to each other.

32. Problem 8. *Draw the two projections of a straight line situated in the second quadrant parallel to H, oblique to V, and 2 units from H.*

33. Problem 9. *Draw the two projections of a straight line situated in the third quadrant perpendicular to H and 3 units from V.*

34. Problem 10. *Given a point M, 6 units back of V and 4 units below H, also a point N, 4 units in front of V and 4 units above H; required to draw the projections of the straight line located by the points M and N.*

35. Problem 11. *Draw the two projections of a straight line situated in the fourth quadrant parallel to H and V and equidistant from H and V.*

36. Problem 12. *Draw the two projections of a straight line situated in the fourth quadrant perpendicular to V and 4 units from H.*

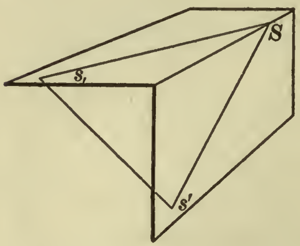

Fig. 23

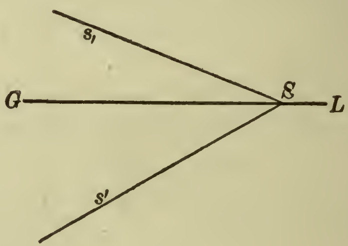

Fig. 24

37. Representation of the Plane. The line in which a plane intersects H is called the *horizontal trace* of the plane. The line in which a plane intersects V is called the *vertical trace* of the plane.

If either trace of a plane intersects G–L, the other trace must intersect G–L at the same point, otherwise G–L would intersect the plane in two points.

If either trace of a plane is parallel to G–L, the other trace must also be parallel to G–L, otherwise G–L would intersect the plane while it was parallel to a straight line in the plane.

As a plane is completely determined either by two intersecting straight lines or by two parallel straight lines, it is evident that a plane is definitely located by its traces.

In Fig. 23, $S\!-\!s_{\prime}$ represents the horizontal trace, and $S\!-\!s'$ represents the vertical trace of a plane S which is oblique to both H and V. The plane is supposed to extend without limit in all directions, and therefore passes through all the quadrants, although in the diagram only that portion of the plane which lies in the third quadrant is represented.

Fig. 24 shows how the traces of the plane in Fig. 23 will appear after H and V take their position of coincidence.

It will be observed that the traces $S\!-\!s_{\prime}$ and $S\!-\!s'$ make the same angle with $G\!-\!L$ after revolution as before, and therefore reveal just as much in regard to the location of the plane.

38. Representation of the Plane by Two Intersecting Straight Lines in Space. Since any two intersecting straight lines or any

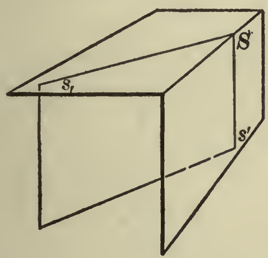

FIG. 25

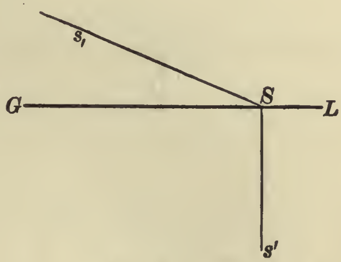

FIG. 26

two parallel straight lines may be used to determine a plane, we may represent a plane by giving the projections of such lines.

By producing these lines until they intersect H and V we shall find points in the horizontal and vertical traces of the plane, if such traces are needed.

39. Representation of Planes occupying Various Positions with Reference to *H* and *V*. Fig. 25 represents a plane in the third quadrant, perpendicular to H but oblique to V. Under these conditions the vertical trace must be perpendicular to $G\!-\!L$ and the horizontal trace must make the same angle with $G\!-\!L$ that the plane makes with V.

Fig. 26 shows how the traces of the plane in Fig. 25 will appear after H and V take their position of coincidence, no change taking place in the relation of the traces to $G\!-\!L$.

Fig. 27 represents a plane in the third quadrant, perpendicular to *V* but oblique to *H*. Under these conditions the horizontal trace must be perpendicular to *G–L* and the vertical trace must make the same angle with *G–L* that the plane makes with *H*.

Fig. 28 shows how the traces of the plane in Fig. 27 will appear after *H* and *V* take their position of coincidence.

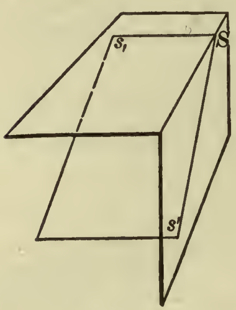

Fig. 27

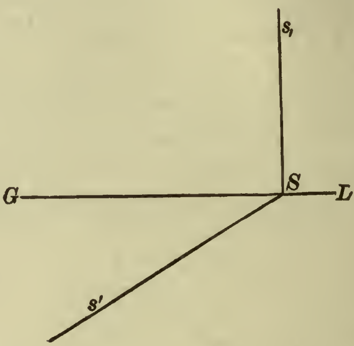

Fig. 28

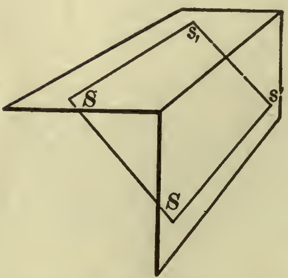

Fig. 29

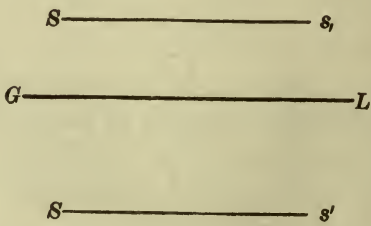

Fig. 30

Fig. 29 represents a plane in the third quadrant and parallel to *G–L*. Its traces then must both be parallel to *G–L*.

Fig. 30 shows how the traces of the plane in Fig. 29 will appear after *H* and *V* take their position of coincidence.

In all the foregoing cases, in order that the diagrams may be more easily understood, only limited portions of the planes have been taken into consideration; but planes must be regarded as

infinite in extent and in no way limited by their traces or by any lines which locate them.

When a plane is oblique to G–L, as the plane S in Figs. 31 and 32, it passes through all the quadrants. Its horizontal trace extends both in front and back of G–L and its vertical trace extends both above and below G–L. The traces S–$s_{,,}$ and S–s'' limit that portion of the plane in the first quadrant, the traces S–$s_{,}$ and S–s'' limit that portion of the plane in the second quadrant, the traces S–$s_{,}$ and S–s' limit that portion of the plane in the third quadrant, and the traces S–$s_{,,}$ and S–s' limit that portion of the plane in the fourth quadrant.

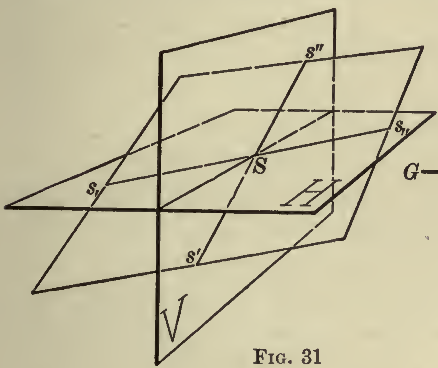

FIG. 31

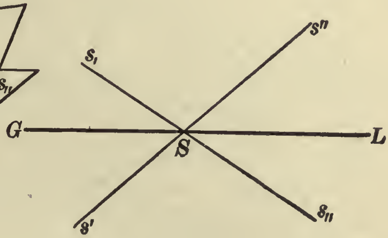

FIG. 32

40. To assume a Plane at Random. Assume the two traces at random so long as they either intersect G–L at the same point or are both parallel to G–L.

41. To assume a Series of Parallel Planes. Since the intersections of a series of parallel planes by an oblique plane is a series of parallel straight lines, it follows that the horizontal traces of parallel planes must be parallel; also that their vertical traces must be parallel.

42. To assume a Straight Line in a Plane.

Principle. If a straight line in a plane pierces H it must pierce it in the horizontal trace of the plane, and if it pierces V it must pierce it in the vertical trace of the plane; for the horizontal trace of the plane is the only line in common between the plane and H, and the vertical trace of the plane is the only line in common between the plane and V.

Construction. In Fig. 33 let S–$s_{,}$ represent the horizontal trace and let S–s' represent the vertical trace of a plane oblique to G–L.

Assume any point, as M, on the horizontal trace. The point $m_{,}$ is the horizontal projection of this point, and the vertical projection must be in G–L at m'.

Assume another point, as N, anywhere in the vertical trace. The point n' is the vertical projection of this point, and the horizontal projection must be in G–L at $n_{,}$.

Since M is a point in the horizontal trace of the plane S, it must be a point in the plane; and since N is a point in the vertical trace of the plane S, it must be a point in the plane. The line M–N is therefore a line of the plane.

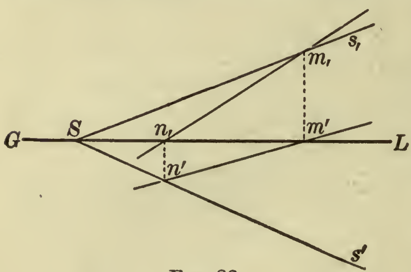

FIG. 33

Principle. If through any point in a plane a straight line is drawn parallel to another straight line already in the plane, the line so drawn is also in the plane.

Principle. If a straight line is in a plane and is parallel to H, it will be parallel to the horizontal trace of the plane, and its horizontal projection will be parallel to the horizontal trace of the plane.

Principle. If a straight line is in a plane and is parallel to V, it will be parallel to the vertical trace of the plane, and its vertical projection will be parallel to the vertical trace of the plane.

43. To assume a Point in a Plane. First, by Section 42, assume a straight line in the plane; then by Section 28 assume a point upon this line.

44. Proposition 1. *If a straight line is perpendicular to a plane, its horizontal projection will be perpendicular to the horizontal trace of the plane, and its vertical projection will be perpendicular to the vertical trace of the plane.*

Proof. Since the line is perpendicular to the given plane, the horizontal projecting plane of the line must be perpendicular to the given plane. The horizontal projecting plane of the line is also perpendicular to H, and therefore since it is perpendicular

both to the given plane and to H, it must be perpendicular to their intersection, which is the horizontal trace of the given plane. Since the horizontal trace of the given plane is perpendicular to the horizontal projecting plane of the given line, it must be perpendicular to any line of the horizontal projecting plane passing through its foot, as the horizontal projection of the given line.

By a similar course of reasoning in connection with the vertical projecting plane of the given line we may prove that the vertical projection of the given line must be perpendicular to the vertical trace of the given plane.

45. Proposition 2. *If the two projections of a straight line are perpendicular respectively to the two traces of a given plane, the line itself will, in general, be perpendicular to the plane.*

Proof. The horizontal projecting plane of the line is perpendicular to the horizontal trace of the given plane and therefore perpendicular to the plane. The vertical projecting plane of the line is perpendicular to the vertical trace of the given plane and therefore perpendicular to the plane. Since the horizontal and vertical projecting planes of the line are both perpendicular to the given plane, their intersection, which is the line itself, must be perpendicular to the plane.

46. Problem 13. *Draw the traces of a plane which is perpendicular to H and makes an angle of* 45 *degrees with V.*

47. Problem 14. *Draw the traces of a plane which is perpendicular to V and makes an angle of* 60 *degrees with H.*

48. Problem 15. *Given a plane which makes an angle of* 30 *degrees with H, and whose horizontal trace is parallel to G–L and* 4 *units back of V; required to draw the vertical trace.*

49. Problem 16. *Draw the traces of a plane parallel to the plane determined in Problem 15.*

50. Problem 17. *Given a plane which is oblique to G–L, whose horizontal trace is back of V, and whose vertical trace is above H; required to draw the projections of a straight line in this plane.*

51. Problem 18. *Given a plane which is parallel to G–L; required to draw the projections of a point in this plane.*

CHAPTER III

SUPPLEMENTARY PLANES OF PROJECTION

52. Supplementary Planes of Projection. While H and V are the principal planes of projection, projection may be made upon any plane which we may choose for that purpose. Any plane other than H or V which is used as a plane of projection is spoken of as a *supplementary plane of projection.*

The most common supplementary plane of projection is a profile plane of projection, — a plane perpendicular to both H and V and consequently perpendicular to G–L.

53. Representation of the Point upon a Profile Plane of Projection. In Fig. 34, which is a pictorial drawing, let the points M, N, O, and P have the same location with reference to H and V as in Fig. 1, and let the plane P represent a profile plane of projection and situated to the right of the given points.

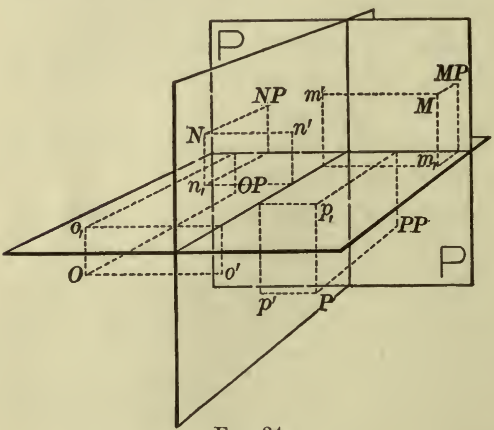

Fig. 34

According to Section 5, the profile projection of M, or the projection of M upon P, is MP; the profile projection of N is NP; the profile projection of O is OP; and the profile projection of P is PP.

Here it will be observed that the distance of the point itself from V is equal to the distance of its profile projection from V, and that the distance of the point itself from H is equal to the distance of its profile projection from H.

It will also be observed that when the projections of a point upon *H*, *V*, and *P* are given, the position of the point with reference to these three planes of projection is definitely determined.

Revolve the plane *P* about its vertical trace as an axis until that portion of the plane *P* in front of *V* falls on *V* to the left of the axis, and that portion of *P* back of *V* falls on *V* to the right of the axis.

Each profile projection will move in a plane parallel to *H*, and in the arc of a circle with center in the axis. *MP* will fall upon *V* at a point as far to the left of the axis as *M* was originally in front of *V*, and as far above *G–L* as *M* was originally above *H*.

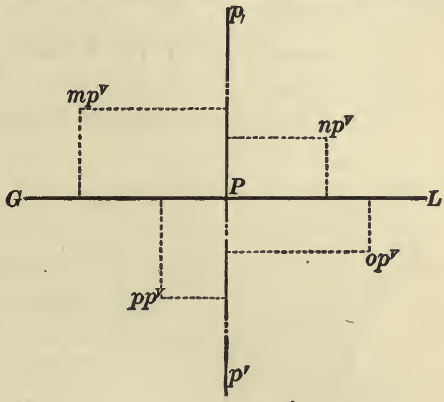

FIG. 35

NP will fall upon *V* at a point as far to the right of the axis as *N* was originally behind *V*, and as far above *G–L* as *N* was originally above *H*. *OP* will fall upon *V* at a point as far to the right of the axis as *O* was originally behind *V*, and as far below *G–L* as *O* was originally below *H*. *PP* will fall upon *V* at a point as far to the left of the axis as the point *P* was originally in front of *V*, and as far below *G–L* as the point *P* was originally below *H*.

After the profile plane has been thus revolved, the profile projections of *M*, *N*, *O*, and *P* will appear as shown in Fig. 35.

This is the picture of the four points which the observer would have if he stood at an infinite distance to the right and looked along visual lines parallel to *G–L*. *H* would appear as a horizontal line corresponding with *G–L* in Fig. 35, and the vertical plane would appear as a vertical line corresponding with $p_{\prime}$–P–p'.

Here points in the first quadrant have their profile projections to the left of the vertical line and above the horizontal line; points in the second quadrant have their profile projections to the right

of the vertical line and above the horizontal line; points in the third quadrant have their profile projections to the right of the vertical line and below the horizontal line; points in the fourth quadrant have their profile projections to the left of the vertical line and below the horizontal line.

Distances of points in front or back of V are indicated by the distances of their profile projections to the left or right of the vertical line. Distances of points above or below H are indicated by the distances of their profile projections above or below the horizontal line.

54. Representation of the Straight Line upon a Profile Plane of Projection. The profile projection of a straight line is determined by the profile projections of two points of the line.

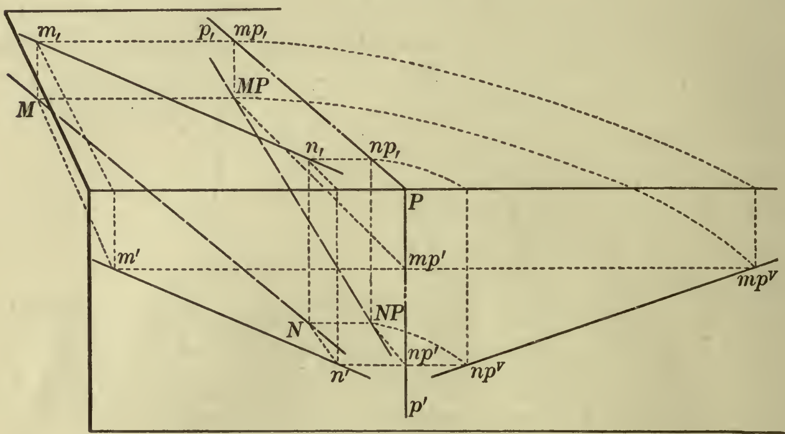

Fig. 36

In Fig. 36, which is a pictorial drawing, let M–N represent the given line and let the plane P represent the profile plane of projection.

The profile projection of M is MP, and the profile projection of N is NP. The line MP–NP is the profile projection of M–N.

Now if we revolve the profile plane of projection about its vertical trace P–p' until it coincides with V, MP will fall at mp^V, and NP will fall at np^V. The line mp^V–np^V is the profile projection of M–N after the revolution.

Fig. 37 shows how the projections of Fig. 36 will appear after the planes H, V, and P have taken their position of coincidence.

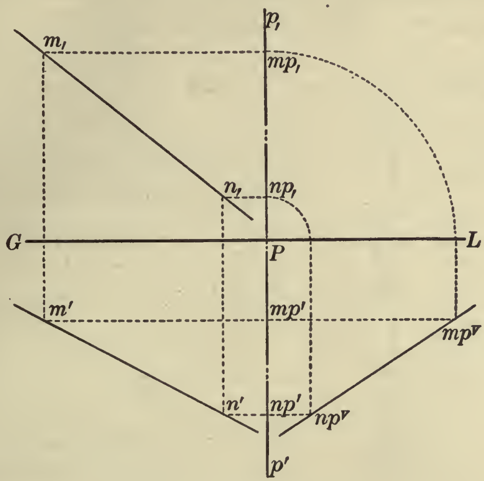

FIG. 37

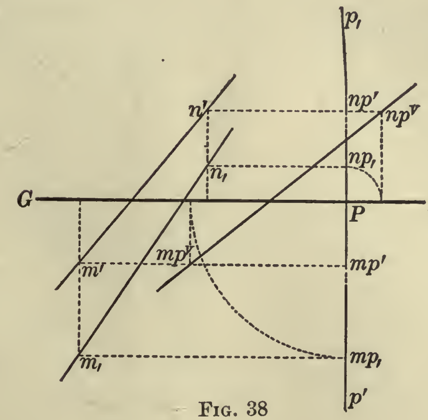

FIG. 38

Fig. 38 shows the projections upon H, V, and P of a straight line M–N running from a point in the fourth quadrant to a point in the second quadrant.

Fig. 39 shows the projections upon H, V, and P of a straight line M–N in a profile plane.

As profile planes are parallel the distance mp^V–np^V in Fig. 39 must be equal to the distance M–N.

55. Other Supplementary Planes of Projection. Supplementary planes of projection may be assumed in any position whatever.

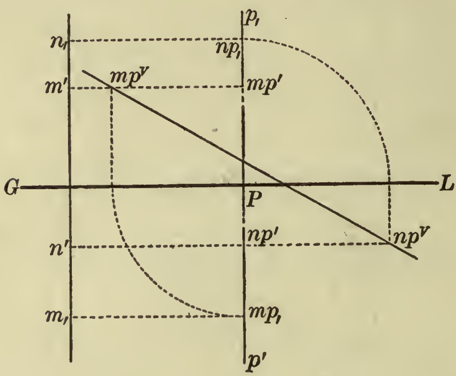

Fig. 39

Fig. 40, which is a pictorial drawing, represents a supplementary plane of projection U, assumed parallel to the horizontal projecting plane of the line M–N.

According to Section 5 the supplementary projection of M in this case is MU and the supplementary projection of N is NU. Therefore the supplementary projection of M–N is MU–NU.

It will be observed that because the supplementary plane of projection is perpendicular to H, the supplementary projection of any point, as M, will fall at the same distance from the horizontal trace U–u_I of the supplementary plane as the point M is from H, or, what is the same thing, as far as the vertical projection m' is from G–L.

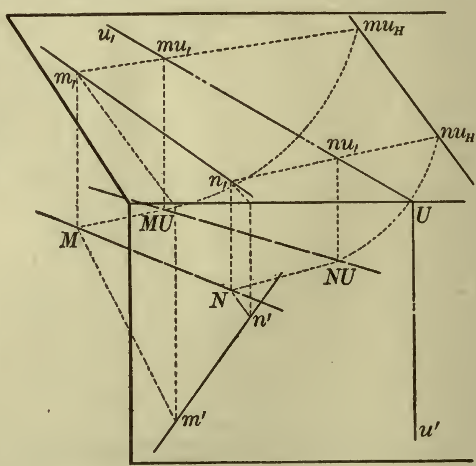

Fig. 40

It will also be observed that the lines MU–mu_I and m_I–mu_I are both perpendicular to U–u_I at the same point.

Now if the supplementary plane U be revolved about its horizontal trace as an axis until it coincides with H, MU will fall at mu_H upon a straight line drawn through m_I

perpendicular to U–$u_{\prime}$ and at a distance from U–$u_{\prime}$ equal to the distance of M from H. For the same reason NU will fall at nu_{H}, and the line mu_{H}–nu_{H} will represent in revolved position the supplementary projection of the line M–N.

Fig. 41 shows how the projections of Fig. 40 will appear after the planes H, V, and U have taken their position of coincidence.

Fig. 42 represents a supplementary plane of projection assumed parallel to the vertical projecting plane of a line M–N. This plane is perpendicular to V and its vertical trace is U–u'. The supplementary projection of M is MU, the supplementary projection of N is NU, and the supplementary projection of the line is MU–NU.

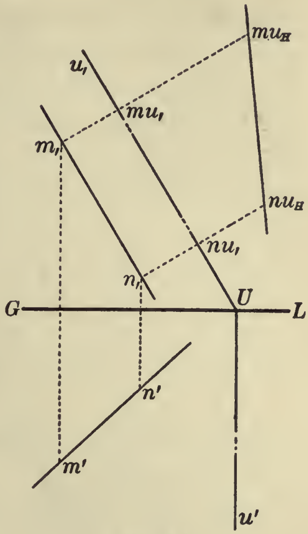

Fig. 41

It will be observed that the supplementary projection of any point, as M, upon this plane is at the same distance from V as the point itself.

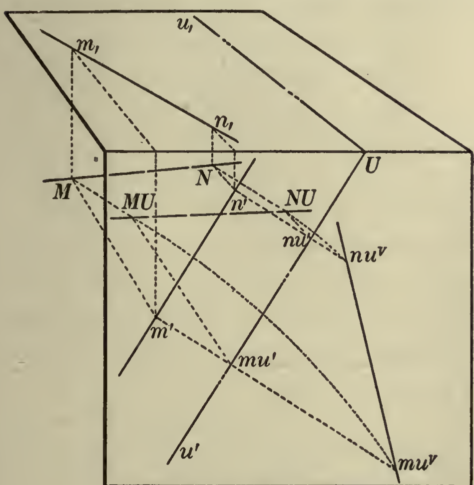

Fig. 42

It will also be observed that the lines MU–mu' and m'–mu' are both perpendicular to U–u' at the same point.

Now if the supplementary plane U be revolved about its vertical trace as an axis until it coincides with V, MU will fall at mu^V upon a straight line drawn through m' perpendicular to U–u' and at a distance from U–u'

equal to the distance of M from V. For the same reason NU will fall at nu^V, and the line mu^V–nu^V will represent in revolved position the supplementary projection of the line M–N.

Fig. 43 shows how the projections of Fig. 42 will appear after the planes H, V, and U have taken their position of coincidence.

56. Revolution of the Supplementary Planes. After projection has been made upon a supplementary plane of projection the plane is revolved about either its horizontal trace or its vertical trace until coincident with the corresponding plane of projection.

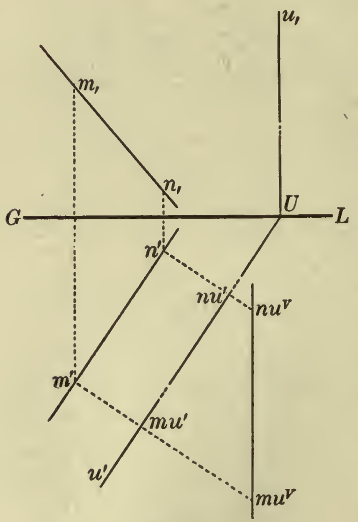

Fig. 43

The revolution is always made in such a way that that portion of the supplementary plane above H or in front of V, according as the axis is in H or in V, shall move toward the principal projections already made upon H or V, and that that portion of the supplementary plane of projection below H or back of V shall move in the opposite direction or away from the principal projections on H or V.

Illustration of this practice may be seen in Fig. 36, where the supplementary projection of M–N upon P back of V is revolved to the right, or away from the projection m'–n'; also in Fig. 38, where that portion of the supplementary plane P in front of V is revolved toward the left and that portion back of V is revolved toward the right, the former toward and the latter away from the projection m'–n'; again in Fig. 40, where that portion of the supplementary plane U below H is revolved toward the right, or away from the projection $m_,$–$n_,$.

In case the supplementary plane is on the left of the magnitude, that portion of the supplementary plane above H or in front of V, according as the axis of revolution is in H or in V, is revolved, according to the rule, toward the right, or in a direction opposite to that assumed by supplementary planes on the right of the magnitude.

57. **Problem 19.** *Given a straight line in the third quadrant and oblique to H and to V; required the supplementary projection of the line upon a profile plane of projection assumed on the left of the line.*

58. **Problem 20.** *Given a straight line in the second quadrant and oblique to H and to V; required the supplementary projection of the line upon a plane parallel to the horizontal projecting plane of the line and assumed on the left of the line.*

59. **Problem 21.** *Given a straight line in the fourth quadrant and oblique to H and to V; required the supplementary projection of the line upon a plane parallel to the vertical projecting plane of the line.*

60. **Problem 22.** *Given a plane parallel to G–L but oblique to H and to V; required the angle which the plane makes with H; also the angle which the plane makes with V. Solve by use of a supplementary profile plane of projection.*

61. **Problem 23.** *Given a plane parallel to G–L but oblique to H and to V; required the traces of a number of planes parallel to the given plane. Solve by use of a supplementary profile plane of projection.*

62. **Problem 24.** *Given a plane parallel to G–L and passing through the fourth, first, and second quadrants; required the traces of a plane parallel to the given plane and passing through the fourth, third, and second quadrants. Solve by use of a supplementary profile plane of projection.*

CHAPTER IV

NOTATION

63. To distinguish between a point in space and its projections on H and V, the point itself will be represented by the capital letter and its projection by the small letter.

64. The horizontal and vertical projections of a point, as M, will be represented by $m_{\prime}$ and m' respectively, and will be read *m sub one* and *m prime one* respectively.

65. If a point, as M, is made to occupy several positions in the same problem, its horizontal and vertical projections will be represented in order by $m_{\prime}$, m'; $m_{\prime\prime}$, m''; $m_{\prime\prime\prime}$, m'''; etc.

66. When two or more points are projected into the same point, the letters of all the points will be written upon this common projection.

67. If a point, as M, is revolved into H, its revolved position will be represented by m_H, and the vertical projection of the point in this revolved position will be represented by m'_H.

68. If a point, as M, is revolved into V, its revolved position will be represented by m^V, and the horizontal projection of the point in this revolved position will be represented by $m_{\prime}^{V}$.

69. The horizontal projection of a vertical projection, as m', will be represented by $(m')_{\prime}$.

70. The vertical projection of a horizontal projection, as $m_{\prime}$, will be represented by $(m_{\prime})'$.

71. If a point, as M, is projected upon a plane, as T, other than H or V, this projection will be represented by MT, and the horizontal and vertical projections of this projection will be represented by $mt_{\prime}$ and mt' respectively.

72. If a point, as M, is projected upon a plane, as P, and if this projection is projected upon another plane, as Q, the last projection will be represented by MPQ, and its horizontal and vertical projections will be represented by $mpq_{\prime}$ and mpq' respectively.

73. If in the last case MPQ should be revolved into H, its revolved position would be represented by mpq_H; if revolved into V, its revolved position would be represented by mpq^V.

74. A new ground line will be represented by either $G_{\prime}-L_{\prime}$ or $G'-L'$, according as it lies in the original H or V.

75. The projections of given or required lines, if visible, will be represented by full lines.

76. The projections of given or required lines, when invisible, and the projections of all construction lines will be represented by broken lines consisting of dashes about $\frac{1}{2}$ inch in length and separated by very small spaces, thus:

77. The traces of given or required planes, if visible, will be represented by full lines.

78. The traces of given or required planes, when invisible, and the traces of all construction planes will be represented by mixed lines, thus:

79. Lines connecting the projections of a point, also the projections of the path in which a point moves, will be represented by dotted lines, thus:

80. The planes H and V and all surfaces and solids used in the construction of a problem will be regarded as transparent.

All surfaces and solids constituting given or required portions of a problem will be regarded as opaque.

CHAPTER V

METHOD OF LOCATING GIVEN PARTS

81. Coördinate Planes of Reference. Conceive a profile plane of projection P perpendicular to G–L, passing through the center of the drawing space and cutting the plane of the drawing in a straight line to be known as the axis of Y.

All distances in space will be referred to the planes P, V, and H, and measured on straight lines perpendicular to them.

Distances measured to the right of P, in front of V and above H, will be considered plus distances.

Distances measured to the left of P, back of V and below H, will be considered minus distances.

82. The Point. A point will be located by giving its distances from P, V, and H. The distance of a point from P is the same as the distance of its horizontal projection from the horizontal trace of the plane P, or as the distance of its vertical projection from the vertical trace of the plane P. The distance of a point from V is the same as the distance of its horizontal projection from G–L. The distance of a point from H is the same as the distance of its vertical projection from G–L.

In giving the position of a point in space its distance from P will be mentioned first, its distance from V second, and its distance from H last. $M = 2, 5, 4$ means that the point M is 2 units to the right of P, 5 units in front of V, and 4 units above H; that the horizontal projection of M is 2 units to the right of Y and 5 units in front of G–L; that the vertical projection of M is 2 units to the right of Y and 4 units above G–L. The letters M, N, O, P, Q, and R will be used in connection with points.

83. The Straight Line. A straight line will be located by giving the position of two of its points. [$M = -2, 2, -5$; $N = 2, -4, 6$] indicates a straight line passing through the points M and N but not necessarily limited by them.

Straight lines will be specified by the letters of the points used in locating them; for example, a straight line passing through the points M and N will be spoken of as the line M–N.

84. The Plane. Planes will be located by their horizontal and vertical traces, that is, by their intersections with H and V respectively.

The traces will be located by giving the position of their vertex, that is, their intersection on G–L, and the angles which they make with G–L.

The angles which the horizontal and vertical traces make with G–L will always be measured, the former clockwise, the latter contra-clockwise, starting from G–L on the right of the vertex.

Of the two angles made with G–L, that made by the horizontal trace will be mentioned first.

$T = 4, 30°, 45°$ means that the vertex is 4 units to the right of Y; that the horizontal trace runs forward toward the right, making an angle of 30 degrees with G–L; and that the vertical trace runs upward toward the right, making an angle of 45 degrees with G–L.

$S = 0°, 4, 3$ indicates that the traces are parallel to G–L; that the horizontal trace is 4 units in front of G–L; and that the vertical trace is 3 units above G–L.

Planes will be specified by the letters S, T, U, and W. The horizontal traces will be represented by S–$s_{\prime}$, T–$t_{\prime}$, U–$u_{\prime}$, W–$w_{\prime}$; and the vertical traces will be represented by S–s', T–t', U–u', and W–w', the capital letter being placed at the vertex.

85. Scale. Problems of this book whose given parts are located with reference to three coördinate planes of projection have been designed to the scale of one quarter inch to the unit, and in the solution of these problems it will be found convenient to make use of this scale.

CHAPTER VI

PROBLEMS RELATING TO THE POINT, LINE, AND PLANE

86. Introductory Statements. As a rule the solution of a problem will be divided into two parts: (1) the analysis, or general theory of solution, in which a clear and logical statement of the method of procedure without reference to any diagram will be made; and (2) the construction, or actual graphic work necessary in the solution, in which the suggestions of the analysis will be followed in order.

The graphic construction of every problem in descriptive geometry should be checked.

A check in drafting is an application of some graphic process by which the accuracy of construction may be tested. No one can be absolutely sure of the accuracy of his results until they have been carefully tested.

The nature of the check for any particular problem may be determined by a study of the various conditions which must be satisfied by the processes of construction.

A problem may be said to be checked when the same result has been obtained by two distinct processes of solution.

87. Problem 25. *Given a straight line in H and a point in space; required to revolve the point about the line as an axis into H.*

Principle. Revolution in descriptive geometry is always made about a rectilinear axis occupying a definite position.

Principle. A point is being revolved about an axis when it moves in a plane perpendicular to the axis and retains a constant distance from the axis. A point situated in the axis will not move during revolution.

Principle. When a plane is revolved about an axis the relative position of magnitudes in the plane is not changed.

Analysis. In Fig. 44, which is a pictorial drawing, let $M–N$ represent the line in H and let O represent the point in space.

Through $o_{\prime}$, the horizontal projection of O, draw $o_{\prime}$–$a_{\prime}$ perpendicular to $m_{\prime}$–$n_{\prime}$. This line must be the horizontal trace of the plane containing O and perpendicular to the axis, and therefore the plane in which O moves during revolution.

After revolution the point O must fall somewhere upon $a_{\prime}$–$o_{\prime}$ produced, and at a distance from the axis equal to its original distance from the axis.

This distance is O–$a_{\prime}$, which is the hypotenuse of the right-angled triangle O–$o_{\prime}$–$a_{\prime}$. The base of this triangle is equal to the distance of the horizontal projection of the point O from the axis, and the altitude is equal to the distance of the point O from H, or, what is the same thing, the distance of the vertical projection of the point O from G–L. Therefore lay off from $a_{\prime}$, upon $a_{\prime}$–$o_{\prime}$

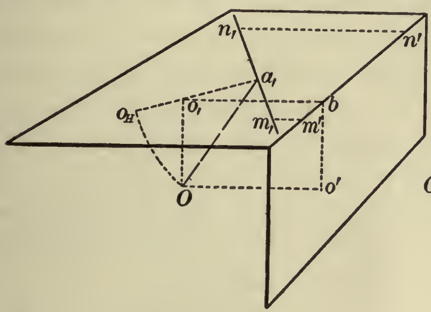

FIG. 44

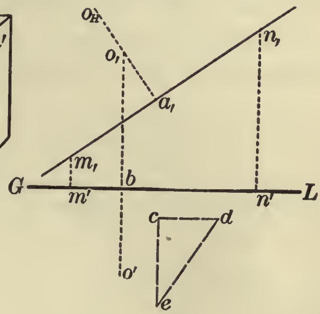

FIG. 45

produced, the distance $a_{\prime}$–o_{II} equal to $a_{\prime}$–O. The point o_{II} will be the revolved position of O.

Construction. Fig. 45 shows how the work is done when the planes H and V occupy their position of coincidence. The line $o_{\prime}$–$a_{\prime}$ is drawn through $o_{\prime}$ perpendicular to $m_{\prime}$–$n_{\prime}$, and $a_{\prime}$–o_{II} is made equal to the hypotenuse of a right-angled triangle whose base is equal to $o_{\prime}$–$a_{\prime}$ and whose altitude is equal to o'–b.

This hypotenuse is calculated graphically in the triangle c–d–e, where c–d equals $o_{\prime}$–$a_{\prime}$ and where c–e equals b–o'.

It will be noticed in Fig. 44 that if the point O is so situated that its horizontal projection falls on $m_{\prime}$–$n_{\prime}$, the base of the triangle will vanish and the hypotenuse will be equal to the altitude. In such a case then the distance of the point from the axis, or the

radius of the arc in which the point moves, will be equal to the distance of the vertical projection of the point from *G–L*.

88. Problem 26. *Given the straight line* [$M = -6, 6, 0$; $N = 6, -2, 0$] *and the point* $O = 0, 6, 5$; *required to revolve O about M–N into H.*

89. Problem 27. *Given the straight line* [$M = -6, -6, 0$; $N = 6, 2, 0$] *and the point* $O = 0, -6, 4$; *required to revolve O about M–N into H.*

90. Problem 28. *Given the straight line* [$M = -6, 2, 0$; $N = -6, -4, 0$] *and the point* $O = 0, 4, -6$; *required to revolve O about M–N into H.*

91. Problem 29. *Given a straight line in V and a point in space; required to revolve the point about the line as an axis into V.*

Analysis. From what has been said in connection with Problem 25, it will be evident that the vertical trace of the plane of revolution must pass through the vertical projection of the given point and must be perpendicular to the given line; also that the radius of the arc in which the point moves will be equal to the hypotenuse of a right-angled triangle whose base is equal to the distance of the vertical projection of the point from the line, and whose altitude is equal to the distance of the horizontal projection of the point from *G–L*.

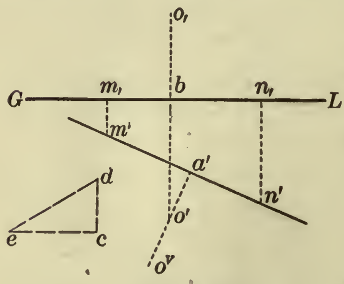

Fig. 46

Construction. In Fig. 46 let *M–N* represent the line in *V* and let *O* represent the point in space. Through o' draw o'–a' perpendicular to m'–n'. This must be the vertical trace of the plane of revolution. Make a'–o^V equal to the hypotenuse of a right-angled triangle *c–d–e*, whose base *c–d* is equal to o'–a' and whose altitude *c–e* is equal to b–$o_{\prime}$.

It will be noticed, as in Problem 25, that if the point *O* is so situated that its vertical projection falls on m'–n', the base of the triangle will vanish and the hypotenuse will be equal to the altitude. In such a case then the radius of the arc of revolution will be equal to the distance of the horizontal projection of the point from *G–L*.

92. Problem 30. *Given the straight line* [$M = -6, 0, -5$; $N = 6, 0, 4$] *and the point* $O = 0, 4, 6$; *required to revolve O about M–N into V.*

93. Problem 31. *Given the straight line* $[M = -6, 0, -5;\ N = 6, 0, -2]$ *and the point* $O = 0, -4, 6$; *required to revolve* O *about* M–N *into* V.

94. Problem 32. *Given the straight line* $[M = 6, 0, 6;\ N = 6, 0, -2]$ *and the point* $O = 0, 4, -6$; *required to revolve* O *about* M–N *into* V.

95. Problem 33. *Given a straight line in space parallel to* H, *also given a point in space; required to revolve the point about the line as an axis until the plane of the point and the line is parallel to* H.

Analysis and Construction. In Fig. 47 let M–N represent the given line and let O represent the given point.

As in Problem 25, $o_{\prime}$–$a_{\prime}$ represents the horizontal trace of the plane containing the point and perpendicular to the axis, and therefore the plane in which O moves during revolution.

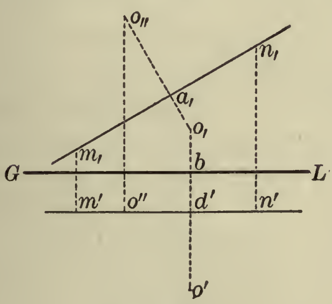

FIG. 47

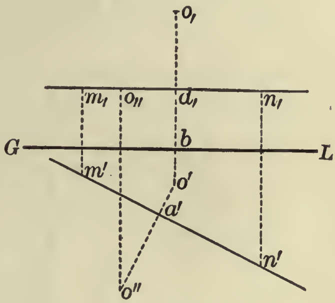

FIG. 48

After revolution the horizontal projection of the point O must fall somewhere upon $o_{\prime}$–$a_{\prime}$ produced, and at a distance from $a_{\prime}$ equal to the hypotenuse of a right-angled triangle whose base is equal to $o_{\prime}$–$a_{\prime}$ and whose altitude is equal to o'–d', since the axis is not in H but at a distance b–d' below H.

The point $o_{\prime\prime}$ represents the horizontal projection of O in the required revolved position, and o'', at the same distance below H as M–N and in a straight line through $o_{\prime\prime}$ perpendicular to G–L, represents the vertical projection of O in this position.

96. Problem 34. *Given a straight line in space parallel to* V, *also given a point in space; required to revolve the point about the line as an axis until the plane of the point and the line is parallel to* V. *See Fig. 48.*

97. Problem 35. *Given a straight line in space parallel to G–L, also given a point in space; required to revolve the point about the line as an axis until the plane of the point and the line is parallel to H, and again until the plane of the point and the line is parallel to V.*

98. Problem 36. *Given a straight line perpendicular to H, also given a point in space; required to revolve the point about the line as an axis.*

Analysis. In Fig. 49, which is a pictorial drawing, let *M–N* represent the line and let O ($o_{\prime}$, o') represent the point.

Since the point O must move in a plane perpendicular to *M–N* and therefore parallel to *H*, and since the point O must retain a

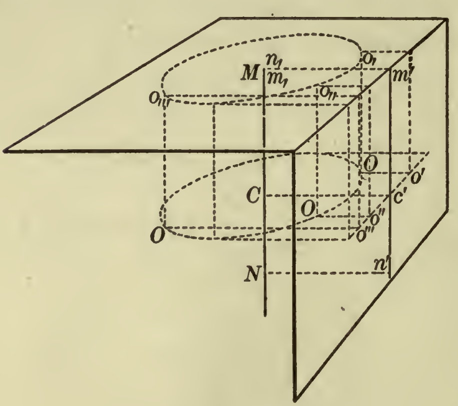

Fig. 49

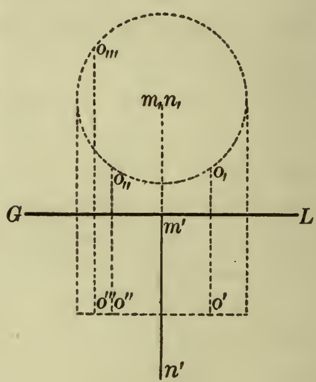

Fig. 50

constant distance from the axis *M–N*, the horizontal projection $o_{\prime}$ of the point must move in the arc of a circle with center at the point where the axis pierces *H*; and its vertical projection o' must move in a straight line o'–o''' drawn through o' parallel to *G–L*.

Now if the point O occupy the several positions indicated in the diagram, the projections will fall at $o_{\prime}$, o'; $o_{\prime\prime}$, o''; $o_{\prime\prime\prime}$, o'''; etc.

Construction. Fig. 50 shows how the work is done when the planes *H* and *V* occupy their position of coincidence. The line m'–n' represents the vertical projection of the given line, and the point $m_{\prime}$ represents the horizontal projection of the same line. The points $o_{\prime}$ and o' represent the two projections of the given point in its first position.

With $m_{\prime}$ as a center and with $m_{\prime}-o_{\prime}$ as a radius draw the circle $o_{\prime}-o_{\prime\prime}-o_{\prime\prime\prime}$. This circle represents the path of the horizontal projection of the point during revolution. Through o' draw the straight line $o'-o'''$ parallel to $G-L$. This line represents the path of the vertical projection of the point during revolution.

Now when O occupies such a position that its horizontal projection falls at $o_{\prime\prime}$, its vertical projection will take the position o'' in a straight line through $o_{\prime\prime}$ perpendicular to $G-L$.

In the same way the two projections of O in any position of the circuit may be found.

99. Problem 37. *Given the straight line* [$M = 0, -6, -6$; $N = 0, -6, -1$] *and the point* $O = -3, 2, 3$; *required to revolve* O *about* $M-N$ *as an axis through arcs of* 30 *degrees and* 160 *degrees.*

100. Problem 38. *Given the straight line* [$M = 0, -2, 6$; $N = 0, -2, 1$] *and the point* $O = 3, 4, -2$; *required to revolve the point* O *about* $M-N$ *as an axis through arcs of* 45 *degrees and* 135 *degrees.*

101. Problem 39. *Given a straight line perpendicular to* V, *also given a point in space; required to revolve the point about the line as an axis.*

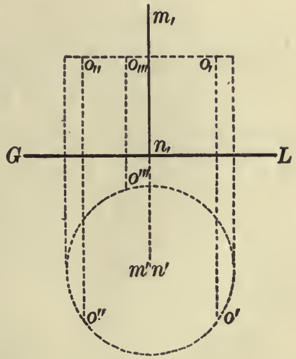

FIG. 51

Analysis and Construction. In Fig. 51 let $M-N$ represent the given line, and let O ($o_{\prime}$, o') represent the given point. Since the point O must move in a plane perpendicular to $M-N$ and therefore parallel to V, and since the point O must retain a constant distance from the axis $M-N$, the vertical projection of the point, namely o', must move in the arc of a circle with center at the point where the axis pierces V; and its horizontal projection $o_{\prime}$ must move in a straight line $o_{\prime}-o_{\prime\prime}$ drawn through $o_{\prime}$ parallel to $G-L$. Now if the point O occupy the several positions indicated in the diagram, the projections will fall at $o_{\prime}$, o'; $o_{\prime\prime}$, o''; $o_{\prime\prime\prime}$, o'''; etc.

102. Problem 40. *Given the straight line* [$M = 0, -1, -6$; $N = 0, -6, -6$] *and the point* $O = -3, -4, 2$; *required to revolve the point* O *about* $M-N$ *as an axis through arcs of* 30 *degrees and* 90 *degrees.*

103. Problem 41. *Given the straight line* $[M = 0, 1, 6; N = 0, 6, 6]$ *and the point* $O = 3, 4, -2$; *required to revolve the point* O *about* M–N *as an axis through arcs of* 60 *degrees and* 90 *degrees.*

104. Problem 42. *Given two intersecting straight lines, one of which is perpendicular to* H *and the other oblique to* H; *required to revolve the latter about the former as an axis.*

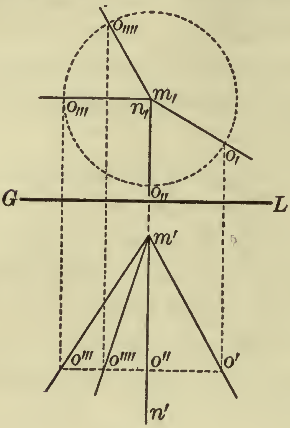

FIG. 52

Analysis and Construction. In Fig. 52 let M–N ($m_{\prime}$–$n_{\prime}$, m'–n') represent the line perpendicular to H, and let M–O ($m_{\prime}$–$o_{\prime}$, m'–o') represent the line oblique to H.

Since the point M, in the line M–O, is also in the axis M–N, it will not move during revolution.

Any other point, as O, in M–O will move as explained in Problem 36.

When the line M–O occupies such a position that its horizontal projection falls at $m_{\prime}$–$o_{\prime\prime}$, the vertical projection of O will fall at o'', and the vertical projection of the line in this position will fall at m'–o''.

In the same way the two projections of M–O in any position of the circuit may be found.

105. Problem 43. *Given two intersecting straight lines one of which is perpendicular to* V *and the other oblique to* V; *required to revolve the latter about the former as an axis.*

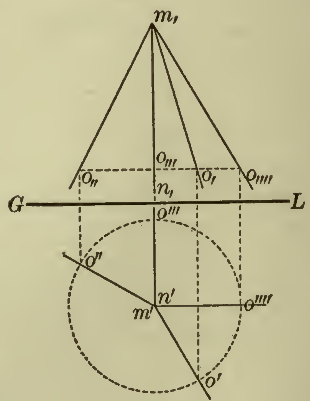

FIG. 53

Analysis and Construction. In Fig. 53 let M–N ($m_{\prime}$–$n_{\prime}$, m'–n') represent the line perpendicular to V, and let M–O ($m_{\prime}$–$o_{\prime}$, m'–o') represent the line oblique to V.

For the same reasons presented in Section 104, M, which is a point in the axis, will remain stationary, and O will move as indicated in the diagram.

106. Problem 44. *Draw the two projections of a straight line passing through the point $M = -4, -4, 2$, running parallel to V, and making an angle of 60 degrees with H.*

107. Problem 45. *Draw the two projections of a straight line passing through the point $M = -4, 3, -1$, running parallel to H, and making an angle of 45 degrees with V.*

108. Problem 46. *Given the straight line [$M = -4, 5, 1$; $N = 4, 2, 6$]; required to find the true distance between M and N, and to determine the angle which the line makes with H.*

Analysis. Revolve the line about an axis through *M* perpendicular to *H* until the line is parallel to *V*. The vertical projection of any portion of the line in this new position will be equal in length to the assumed portion of the line itself, and the angle which this vertical projection of the line in this new position makes with *G–L* will indicate the angle which the line itself in true position makes with *H*.

109. Problem 47. *Given the straight line [$M = -4, 5, 1$; $N = 4, 2, 6$]; required to find the true distance between the points M and N, and to determine the angle which the line makes with V.*

Analysis. Revolve the line about an axis through *M* perpendicular to *V* until the line is parallel to *H*. The horizontal projection of any portion of the line in this new position will be equal in length to the assumed portion of the line itself, and the angle which this horizontal projection makes with *G–L* will indicate the angle which the line itself in true position makes with *V*.

110. Problem 48. *Draw the two projections of a straight line 5 units long, passing through the point $M = 2, -4, -1$, making an angle of 30 degrees with H, and in such a position that its horizontal projection makes an angle of 45 degrees with G–L.*

Analysis. First draw the projections of the line when it passes through the given point, is parallel to *V*, and makes the required angle with *H*. Then revolve the line about an axis through the given point perpendicular to *H* until the horizontal projection of the line makes the required angle with *G–L*. The two projections of the line in this last position will be the required projections.

Construction. See Fig. 54. The projections of the point M are $m_{,}$, m'. The projections of the line, when the line is parallel to V and makes an angle of 30 degrees with H, are $m_{,}$–$n_{,,}$ and m'–n'', where $m_{,}$–$n_{,,}$ is parallel to G–L and where m'–n'' is 5 units long and makes an angle of 30 degrees with G–L.

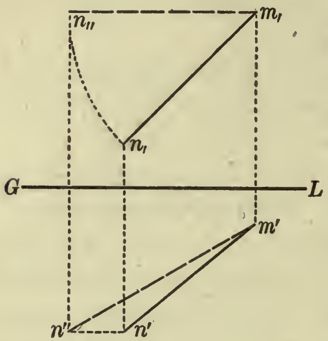

FIG. 54

When the line is revolved to its final position, $n_{,,}$ will move in an arc of a circle to $n_{,}$, and n'' will move in a straight line parallel to G–L, to n'. The required projections are then $m_{,}$–$n_{,}$ and m'–n'.

The line just located is in the third quadrant and runs from the given point downward to the left and toward V.

Consider other positions which the line may occupy and yet satisfy the requirements of the problem.

111. Problem 49. *Construct Problem 48 when* $M = -2, -5, 1$.

112. Problem 50. *Construct Problem 48 when* $M = 2, 5, 1$.

113. Problem 51. *Construct Problem 48 when* $M = 2, 2, -6$.

114. Problem 52. *Draw the two projections of a straight line 4 units long, passing through the point* $M = 1, -1, -3$, *making an angle of 45 degrees with V, and in such a position that its vertical projection makes an angle of 60 degrees with G–L.*

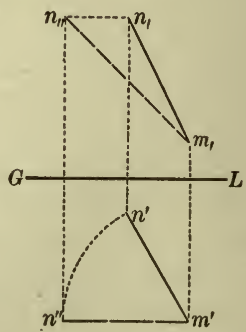

FIG. 55

Analysis. First draw the projections of the line when it passes through the given point, is parallel to H, and makes the required angle with V. Then revolve the line about an axis through the given point perpendicular to V, until the vertical projection of the line makes the required angle with G–L. The two projections of the line in this last position will be the required projections.

Construction. See Fig. 55. The projections of the given point M are $m_{,}$, m'. The projections of the line, when the line is parallel

to H and makes an angle of 45 degrees with V, are $m_{\prime}$–$n_{\prime\prime}$ and m'–n'', where $m_{\prime}$–$n_{\prime\prime}$ is 4 units long and makes an angle of 45 degrees with G–L, and where m'–n'' is parallel to G–L.

When the line is revolved to its final position, n'' will move in the arc of a circle to n', and $n_{\prime\prime}$ will move in a straight line parallel to G–L, to $n_{\prime}$. The required projections are then $m_{\prime}$–$n_{\prime}$ and m'–n'.

The line just located is in the third quadrant and runs from the given point upward to the left and away from V.

Consider other positions which the line may occupy and yet satisfy the requirements of the problem.

115. Problem 53. *Construct Problem 52 when* $M = -3, 6, -2$.

116. Problem 54. *Construct Problem 52 when* $M = 4, 6, 4$.

117. Problem 55. *Construct Problem 52 when* $M = -4, -6, 2$.

118. Problem 56. *Draw the two projections of a straight line* 6 *units long, passing through a point* $M = 0, -3, -1$, *making an angle of* 60 *degrees with* H, *and lying in a profile plane.*

Analysis. First draw the projections of the line when it passes through the given point, is parallel to V, and makes the required angle with H. Then revolve the line about an axis through the given point perpendicular to H until the horizontal projection of the line makes the proper angle with G–L (see Section 104).

119. Problem 57. *Given a rectangular card whose dimensions are* 6 *units by* 4 *units, whose surface is parallel to* V, *whose long edges are perpendicular to* H, *and whose upper right-hand vertex is a point* $B = 2, -1, -1$; *required (1) to draw the two projections of the card in the given position, and (2) to revolve the card about its right-hand vertical edge as an axis, through angles of* 30 *degrees,* 45 *degrees,* 60 *degrees, and* 90 *degrees, and to draw the corresponding projections.*

Construction. See Fig. 56. The projections of the card in its first position are $a_{\prime}$–$b_{\prime}$ and a'–b'–d'–e', where $b_{\prime}$ and b' are the projections of B, where a'–b' is 4 units long, and where a'–e' is 6 units long.

When the card is revolved about B–D as an axis through an angle of 30 degrees, the two projections of the card are $a_{\prime\prime}$–$b_{\prime}$ and a''–b'–d'–e'', where the angle $a_{\prime}$–$b_{\prime}$–$a_{\prime\prime}$ is 30 degrees, and where a'' and e'' are in a straight line through $a_{\prime\prime}$ perpendicular to G–L.

When the card is revolved through an angle of 45 degrees, its two projections will be $a_{\prime\prime\prime}-b_{\prime}$ and $a'''-b'-d'-e.'''$

The projections of the card in the different positions are easily determined when we remember that horizontal projections will move in arcs of circles and that vertical projections will move in straight lines parallel to $G-L$ (see Section 98).

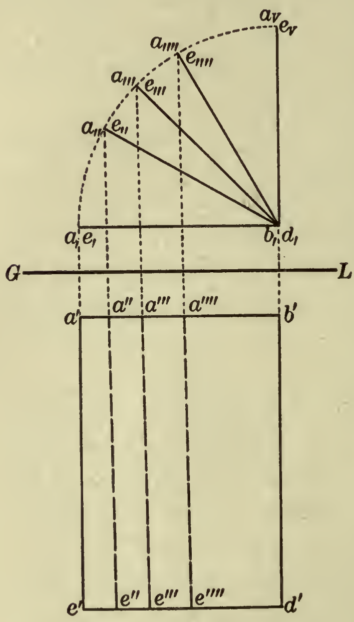

FIG. 56

120. Problem 58. *Given a rectangular card situated in the third quadrant, whose dimensions are* 6 *units by* 4 *units, whose surface is parallel to H, and whose long edges are perpendicular to V; required (1) to draw the projections of the card in the given position, and (2) to revolve the card about its left-hand edge as an axis, through angles of* 30 *degrees,* 45 *degrees,* 60 *degrees, and* 90 *degrees, and to draw the corresponding projections.*

121. Problem 59. *Given a rectangular card situated in the first quadrant, whose dimensions are* 6 *units by* 4 *units, whose surface is parallel to V, and whose long edges are perpendicular to H; required (1) to draw the projections of the card in the given position, and (2) to revolve the card about one of its vertical edges as an axis, through angles of* 30 *degrees,* 45 *degrees,* 60 *degrees, and* 90 *degrees, and to draw the corresponding projections.*

122. Problem 60. *Given a rectangular card whose dimensions are* 5 *units by* 3 *units, whose surface is parallel to H, whose long edges are parallel to V, and whose left-hand back vertex is* $A = -3, -4, -1$; *required (1) to draw the projections of the card in the given position; (2) to revolve the card about its left-hand edge as an axis until the surface of the card is inclined* 30 *degrees to H, and to draw the corresponding projections; and (3) to revolve the card in its last position about a vertical axis through A until the horizontal projections of the*

edges of the card are inclined 45 *degrees to G–L, and to draw the corresponding projections.*

Construction. See Fig. 57. The projections of the card in its first position are $a_{\prime}$–$b_{\prime}$–$d_{\prime}$–$e_{\prime}$ and a'–b'–d'–e'.

In taking its second position the card is revolved about an axis through *A* perpendicular to *V*. The surface of the card will remain perpendicular to *V* and its vertical projection will take the position a'–b'' equal to a'–b', and making an angle of 30 degrees with a'–b'. Its horizontal projection will be $a_{\prime}$–$b_{\prime\prime}$–$d_{\prime\prime}$–$e_{\prime}$, where *A* and *E* have made no change in position and where $b_{\prime\prime}$ and $d_{\prime\prime}$ fall in a straight line through b'' perpendicular to *G–L*.

In taking its third position the card is revolved about an axis through *A* perpendicular to *H*. The horizontal projection of the card will change in position but not in character.

Upon $a_{\prime}$, which makes no change in position, construct the rectangle $a_{\prime}$–$b_{\prime\prime\prime}$–$d_{\prime\prime\prime}$–$e_{\prime\prime}$ equal in every respect to $a_{\prime}$–$b_{\prime\prime}$–$d_{\prime\prime}$–$e_{\prime}$ but having its edges inclined 45 degrees to *G–L*. This is the horizontal projection of the card in its final position.

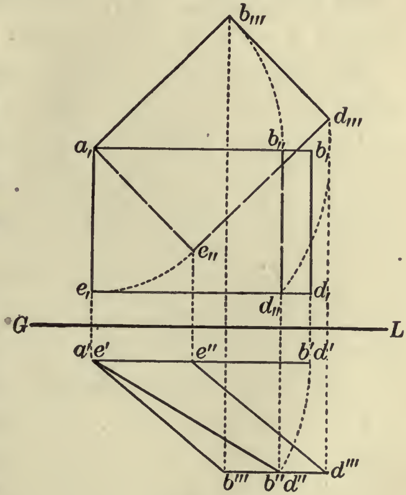

Fig. 57

The vertical projection of *A* will remain at a'. The vertical projection of *B* must be in a straight line through b'' parallel to *G–L*, and also in a straight line through $b_{\prime\prime\prime}$ perpendicular to *G–L*, or at b'''.

The vertical projection of *D* must be in a straight line through d'' parallel to *G–L*, and also in a straight line through $d_{\prime\prime\prime}$ perpendicular to *G–L*, or at d'''.

The vertical projection of *E* must be upon a straight line through e' parallel to *G–L*, and also upon a straight line through $e_{\prime\prime}$ perpendicular to *G–L*, or at e''.

The vertical projection of the card in its final position is a'–b'''–d'''–e''.

123. Problem 61. *Given a rectangular card whose dimensions are* 6 *units by* 4 *units, whose surface is parallel to V, whose long edges are parallel to H, and whose right-hand upper vertex is* $E = 4, -2, -1$; *required (1) to draw the projections of the card in the given position; (2) to revolve the card about its right-hand vertical edge as an axis until the surface of the card is inclined* 30 *degrees to V, and to draw the corresponding projections; and (3) to revolve the card in its last position about an axis through E perpendicular to V until the vertical projections of the edges of the card are inclined* 45 *degrees to G–L, and to draw the corresponding projections.*

124. Problem 62. *Solve Problem 60, substituting a new value for A, namely* $A = -4, 1, 1$, *locating the whole work in the first quadrant.*

125. Problem 63. *Given an hexagonal card whose side is* 3 *units, whose surface is parallel to V, two of whose sides are parallel to H, and whose extreme left-hand vertex is* $A = -4, -1, -4$; *required (1) to draw the projections of the card in the given position; (2) to revolve the card about a vertical axis through A until the surface of the card makes an angle of* 60 *degrees with V, and to draw the corresponding projections; and (3) to revolve the card in its last position about an axis through A perpendicular to V until the vertical projections of the sides which were parallel to H in the last position shall make an angle of* 45 *degrees with G–L, and to draw the corresponding projections.*

Construction. See Fig. 58. The projections of the card in its first position are $a_{\prime}–b_{\prime}–d_{\prime}–e_{\prime}–f_{\prime}–g_{\prime}$ and $a'–b'–d'–e'–f'–g'$.

In taking its second position the card is revolved about a vertical axis through A. The surface of the card will remain perpendicular to H and its horizontal projection will take the position $a_{\prime}–e_{\prime\prime}$ equal to $a_{\prime}–e_{\prime}$, and making an angle of 60 degrees with $a_{\prime}–e_{\prime}$. Its vertical projection will be $a'–b''–d''–e''–f''–g''$, where a' has made no change in position, where b'' is at the intersection of a straight line through b' parallel to $G–L$, with a straight line through $b_{\prime\prime}$ perpendicular to $G–L$, where d'' is at the intersection of a straight line through d' parallel to $G–L$, with a straight line through $d_{\prime\prime}$ perpendicular to $G–L$, etc.

In taking its third position the card is revolved about an axis through A perpendicular to V. The vertical projection will change in position but not in character.

Upon a', which makes no change in position, construct the polygon a'–b'''–d'''–e'''–f'''–g''' equal in every respect to a'–b''–d''–e''–f''–g'', but having the sides which were originally parallel to G–L now inclined at an angle of 45 degrees to G–L. This is the vertical projection of the card in its final position.

The horizontal projection of A will remain at $a_{,}$.

The horizontal projection of B must be in a straight line through $b_{,,}$ parallel to G–L, and also in a straight line through b''' perpendicular to G–L, or at $b_{,,,}$.

The horizontal projection of D must be in a straight line through $d_{,,}$ parallel to G–L, and also in a straight line through d''' perpendicular to G–L, or at $d_{,,,}$.

The horizontal projection of E must be in a straight line through $e_{,,}$ parallel to G–L, and also in a straight line through e''' perpendicular to G–L, or at $e_{,,,}$.

The horizontal projection of the card in its final position is $a_{,}$–$b_{,,,}$–$d_{,,,}$–$e_{,,,}$–$f_{,,,}$–$g_{,,,}$.

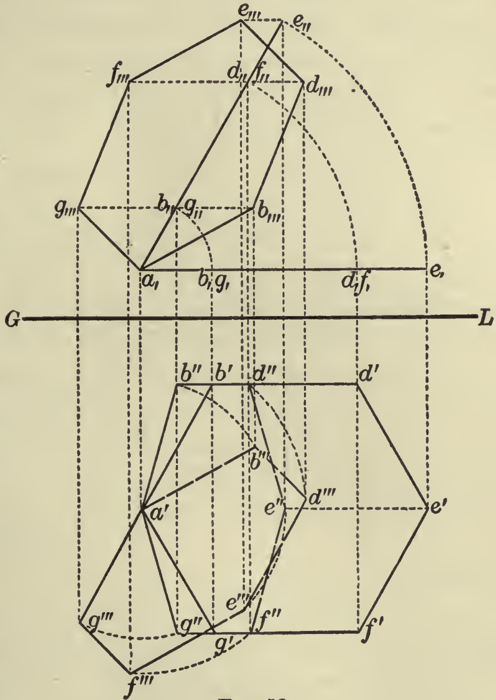

FIG. 58

126. Problem 64. *Given an hexagonal card whose side is* 3 *units, whose surface is parallel to H, two of whose sides are parallel to V, and whose extreme left-hand vertex is* $A = -4, -4, -1$; *required (1) to draw the projections of the card in the given position; (2) to revolve the card about an axis through A perpendicular to V until the surface of the card makes an angle of* 60 *degrees with H, and to draw the corresponding projections; and (3) to revolve the card in its last position about an axis through A perpendicular to H until*

the horizontal projections of the sides which were parallel to V in the last position shall be inclined 45 *degrees to G–L, and to draw the corresponding projections.*

127. Problem 65. *Solve Problem 63, substituting a new value for A, namely* $A = -4, 1, 4$, *locating the whole work in the first quadrant.*

128. Problem 66. *To find the points in which a given straight line intersects H and V.*

CASE 1. *To find the point in which the line intersects H.*

Analysis. Since the point in which the line intersects H must be both in H and in the line itself, its vertical projection must be both in G–L and in the vertical projection of the line, and therefore at their intersection. The horizontal projection of the required

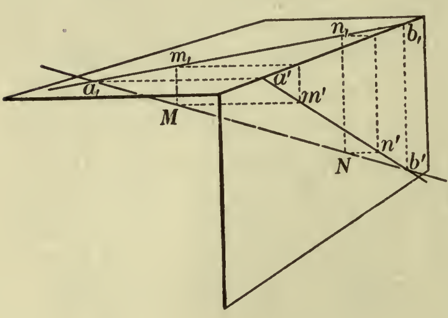

FIG. 59

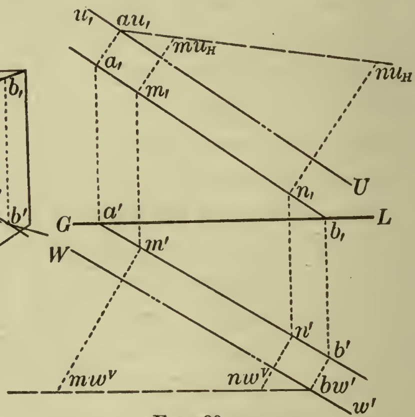

FIG. 60

point must be both in a straight line perpendicular to G–L through the vertical projection just found and also in the horizontal projection of the line, and therefore at their intersection.

Since the required point is in H, it must be coincident with its horizontal projection just found.

Construction. See Figs. 59 and 60. Let M–N represent the given line.

Produce the vertical projection m'–n' to meet G–L at a'. The point a' is by analysis the vertical projection of the required point. At a' draw a straight line perpendicular to G–L to meet the horizontal projection of the line at $a_{,}$, which is both the horizontal

projection of the point and the point itself in which $M–N$ intersects H.

Check. See Fig. 60. Project $M–N$ upon the supplementary plane U, which is assumed parallel to the horizontal projecting plane of $M–N$.

According to Section 55 the line $mu_{II}–nu_{II}$ is the supplementary projection of $M–N$. Produce $mu_{II}–nu_{II}$ to meet the horizontal trace $U–u_{I}$ in au_{I}, which is the supplementary projection of the point in which $M–N$ intersects H. According to Section 55 the points a_{I} and au_{I} should be in the same straight line perpendicular to $U–u_{I}$.

CASE 2. *To find the point in which the line intersects V.*

Analysis. Since the point in which the line intersects V must be both in V and in the line itself, its horizontal projection must be both in $G–L$ and in the horizontal projection of the line, and therefore at their intersection. The vertical projection of the required point must be both in a straight line perpendicular to $G–L$ through the horizontal projection just found, and also in the vertical projection of the line, and therefore at their intersection.

Since the required point is in V, it must be coincident with its vertical projection just found.

Construction. See Figs. 59 and 60. Produce the horizontal projection of the line to meet $G–L$ at b_{I}. At this point draw a straight line perpendicular to $G–L$ to meet the vertical projection of the line at b', which is the required point.

Check. See Fig. 60. Project $M–N$ upon the supplementary plane W, which is assumed parallel to the vertical projecting plane of $M–N$, and proceed as in Case 1.

129. Problem 67. *Given the straight line* [$M = -6, -6, 2$; $N = 6, 2, -6$]; *required to find the points in which M–N intersects H and V.*

130. Problem 68. *Given the straight line* [$M = -6, -4, 2$; $N = 6, 2, 2$]; *required to find the point in which M–N intersects V.*

131. Problem 69. *Given the straight line* [$M = 6, 2, -6$; $N = -6, 2, 3$]; *required to find the point in which M–N intersects H.*

132. Problem 70. *Given the straight line* [$M = -6, 6, -6$; $N = 6, -6, 6$]; *required to find the points in which M–N intersects H and V.*

133. Problem 71. *Given the straight line* $[M = 0, -6, -2; N = 0, -2, -6]$; *required to find the points in which* M–N *intersects* H *and* V.

134. Problem 72. *To find the distance between two given points in space.*

This distance will be measured on the straight line joining the given points.

Analysis 1. If the straight line connecting the two points be revolved about its horizontal projection as an axis into H, the line will be seen in its true length (see Sections 26 and 87).

Analysis 2. If the straight line connecting the two points be revolved about its vertical projection as an axis into V, the line will be seen in its true length.

Analysis 3. If the straight line connecting the two points be revolved about the horizontal projecting line of one of its points as an axis until the line is parallel to V, the vertical projection of the line in this revolved position will be equal in length to the line itself (see Sections 26 and 104).

Analysis 4. If the straight line connecting the two points be revolved about the vertical projecting line of one of its points as an axis until the line is parallel to H, the horizontal projection of the line in this revolved position will be equal in length to the line itself.

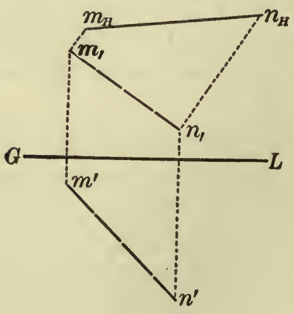

Fig. 61

Construction 1. See Fig. 61. Let M and N represent the two given points. Then $m_{\prime}$–$n_{\prime}$ will represent the horizontal projection and m'–n' will represent the vertical projection of the straight line connecting these points. Following Analysis 1, revolve M–N about $m_{\prime}$–$n_{\prime}$ as an axis into H (see Section 87). M falls at m_H and N falls at n_H, and $m_H - n_H$ is the distance sought.

Construction 2. See Fig. 62. Let M and N represent the given points. Following Analysis 4, revolve M–N about the vertical projecting line of M until M–N is parallel to H (see Section 105). M will remain stationary, but N will move, its vertical projection

taking the position n'' and its horizontal projection taking the position $n_{,,}$. The distance $m_,–n_{,,}$ is the distance sought.

Check. The result obtained by use of any one of the methods here suggested may be checked by using one of the other methods.

135. Problem 73. *Find the distance between the two points $M = -6, -6, 2$ and $N = 6, 2, -4$ by Analysis 2.*

136. Problem 74. *Find the distance between the two points $M = -6, 6, 2$ and $N = 4, -4, -4$ by Analysis 3.*

137. Problem 75. *Find the distance between the two points $M = 0, -6, 1$ and $N = 0, 2, 8$ by any method, and check it by another.*

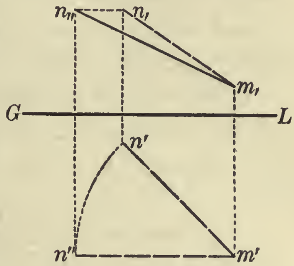

FIG. 62

138. Problem 76. *To pass a plane through three given points.*

Analysis. If we connect any two of the given points by a straight line, such a line must lie in the required plane and must pierce H and V in the corresponding traces of the plane (see Section 42). A straight line connecting either one of the points just used with the remaining point must also lie in the required plane and therefore must pierce H and V in the corresponding traces of the required plane.

A straight line obtained by joining any one of the three given points with any point in a straight line connecting the other two, or by drawing through any one of the three points a straight line parallel to the straight line connecting the other two, is a line of the required plane.

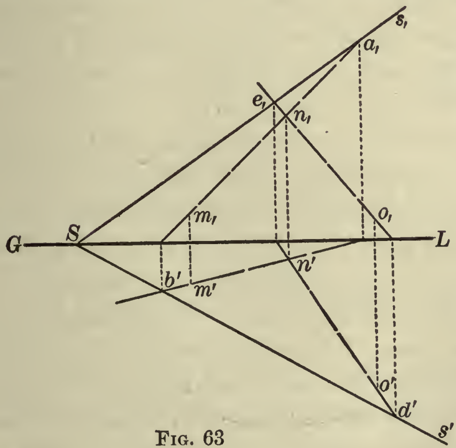

FIG. 63

Construction. See Fig. 63. Let M, N, and O represent the three given points. Connect M and N by a straight line and produce it to intersect H in $a_,$ and V in b'. The point $a_,$ is a point in the

horizontal trace of the required plane, and the point b' is a point in the vertical trace. Connect N and the remaining point O by a straight line. The point $e_{\prime}$ in which N–O intersects H is another point in the horizontal trace, and the point d' in which N–O intersects V is another point in the vertical trace. S–$s_{\prime}$ drawn through $a_{\prime}$ and $e_{\prime}$ is the horizontal trace of the required plane, and S–s' drawn through b' and d' is the vertical trace.

Check. The two traces should intersect in G–L.

139. Problem 77. *Determine the traces of a plane containing the straight line* $[M = -4, 6, 6; N = 2, -2, 1]$ *and the point* $O = 4, 2, -3$.

140. Problem 78. *Determine the traces of a plane containing the two lines* $[M = -6, -6, 4; N = 2, 2, -4]$ *and* $[O = -2, -7, 4; P = 6, 1, -4]$.

141. Problem 79. *Determine the traces of a plane containing the two lines* $[M = -4, 1, 4; N = 4, 1, 4]$ *and* $[O = -6, 4, 1; P = 6, 4, 1]$.

142. Problem 80. *To find the angle between two intersecting straight lines, and to bisect the angle.*

Analysis 1. Revolve the plane of the angle about its horizontal trace or its vertical trace into the corresponding plane of projection. The angle between the two lines will then be seen in its true size (see Section 87), and may be bisected.

Analysis 2. Revolve the plane of the angle about some straight line which is in the plane and parallel to one of the planes of projection, until the plane is parallel to that plane of projection. The projection of the angle upon this plane of projection will then be equal to the angle itself and may be measured and bisected.

Construction. See Fig. 64. Let M–N and N–O, intersecting at N, represent the two given lines. Following Analysis 1, produce M–N to meet H at $a_{\prime}$ and produce N–O to meet H at $b_{\prime}$. The line $s_{\prime\prime}$–S–$s_{\prime}$ through $a_{\prime}$ and $b_{\prime}$ is the horizontal trace of the plane of the given angle. Revolve this plane about $s_{\prime\prime}$–S–$s_{\prime}$ as an axis into H. The points $a_{\prime}$ and $b_{\prime}$, which are points in the sides of the angle and also in the axis, will remain stationary. The vertex N will fall at n_{H} (see Section 87). The angle $a_{\prime}$–n_{H}–$b_{\prime}$ is the required angle. Bisect this angle by the line n_{H}–$d_{\prime}$ intersecting $s_{\prime\prime}$–S–$s_{\prime}$ at $d_{\prime}$.

When the plane of the angle is revolved back to its original position, $d_{,}$ will remain stationary and N will take its old position. The horizontal projection of the bisector in true position will be at $n_{,}-d_{,}$ and its vertical projection will be at $n'-d'$.

Check. Revolve the plane of the angle about its vertical trace into the vertical plane of projection.

143. Problem 81. *Determine the angle between the two lines* $[M = -6, 4, -2; N = 0, -4, -6]$ *and* $[N = 0, -4, -6; O = 6, 2, 1]$.

144. Problem 82. *Determine the angle between the two lines* $[M = -4, 5, 6; N = 4, 1, 2]$ *and* $[N = 4, 1, 2; O = 4, 6, 2]$.

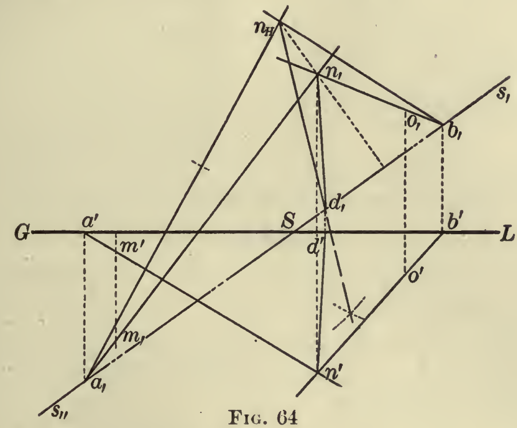

FIG. 64

145. Problem 83. *Determine the angle between the two lines* $[M = -4, -6, -6; N = 4, -2, -2]$ *and* $[N = 4, -2, -2; O = 4, -2, -5]$.

146. Problem 84. *To find the intersection of two planes when their traces are known.*

CASE 1. *When the horizontal traces of the two planes intersect within the limits of the drawing and when the vertical traces of the same planes also intersect within the limits of the drawing.*

Analysis. The intersection of the horizontal traces must be a point common to both planes and therefore a point in their intersection. For the same reason the intersection of the vertical traces must be another point in the required intersection.

Since the intersection of two planes is a straight line, the straight line joining these two points must be the required line of intersection.

Construction. See Fig. 65. Let S and T represent the two given planes. The horizontal traces intersect at $a_{,}$ vertically projected at a'.

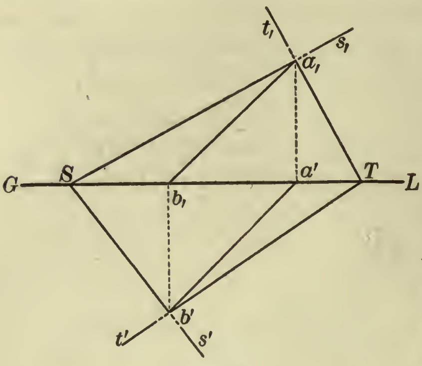

FIG. 65

The vertical traces intersect at b' horizontally projected at $b_{,}$. According to analysis A–B is the required intersection.

Check. Connect any point in the line of intersection with any point in the vertical trace of one of the planes, and note whether this line intersects H in the horizontal trace of the same plane. Apply the same test with reference to the other plane.

CASE 2. *When either the two horizontal traces or the two vertical traces do not intersect within the limits of the drawing.*

Analysis. Pass a series of auxiliary planes parallel to the plane of projection on which the traces intersect. These planes will cut from the two given planes straight lines which will be parallel to this plane of projection and which will intersect in points common to both of the given planes and therefore in their line of intersection.

Construction. See Fig. 66. Let S and T represent the given planes whose horizontal traces do not intersect within the limits of the drawing. The vertical traces intersect at a' horizontally projected at $a_{,}$, locating one point in the required intersection. Pass an auxiliary plane U

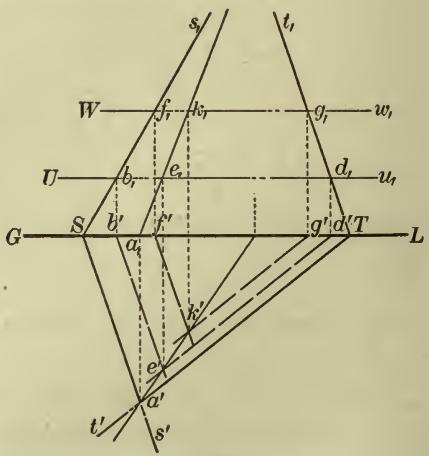

FIG. 66

parallel to V. The plane U cuts the plane S in a straight line B–E parallel to S–s' and vertically projected in b'–e' parallel to S–s' (see Sections 30 and 41). The plane U cuts the plane T in the line D–E vertically projected in d'–e'. The point e' in which the two vertical projections intersect must be the vertical projection of the point in which the lines B–E and D–E intersect, and is, according to analysis, the vertical projection of a point in the required intersection. Since the point E is in the plane U, its horizontal projection $e_{,}$ must be upon U–$u_{,}$, and therefore at the intersection of U–$u_{,}$ and a straight line through e' perpendicular to G–L.

Pass another auxiliary plane W parallel to V, and in the same manner obtain K, another point in the required intersection.

The line A–E–K, whose horizontal projection is $a_{,}$–$e_{,}$–$k_{,}$ and whose vertical projection is a'–e'–k', is the required line.

Check. The straight line E–K thus determined by the two points E and K should pass through A.

147. Problem 85. *Find the intersection of the two planes* $S = -2$, $330°$, $300°$, *and* $T = 0°$, -3, ∞.

148. Problem 86. *Find the intersection of the two planes* $S = -2$, $60°$, $30°$, *and* $T = 0°$, ∞, 2.

149. Problem 87. *Find the intersection of the two planes* $S = -6$, $30°$, $90°$, *and* $T = 6$, $135°$, $90°$.

150. Problem 88. *Find the intersection of the two planes* $S = 0°$, 6, 2, *and* $T = 0°$, 2, 6.

151. Problem 89. *To find the point in which a given straight line intersects a given plane.*

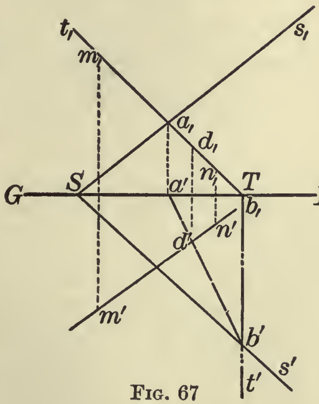

FIG. 67

Analysis. The given line must intersect the given plane in the line in which any auxiliary plane containing the given line intersects the given plane. The point in which the given line crosses this line of intersection must be the required point.

CASE 1. *When the given plane is located by its traces.*

Construction. See Fig. 67. Let M–N represent the given line and let S represent the given plane.

For ease of construction let us take the horizontal projecting plane of *M–N* as the auxiliary plane. This plane *T* intersects *S* in *A–B* (see Section 146), which is cut by *M–N* at the required point *D*.

Check. Use the vertical projecting plane of *M–N*.

CASE 2. *When the given plane is located by two intersecting straight lines.*

Construction. See Fig. 68. Let *M–N* and *O–P*, intersecting at *L*, locate the given plane, and let *Q–R* represent the intersecting line.

By Case 1 above, *M–N* intersects the horizontal projecting plane of *Q–R* at *A*. *O–P* intersects the same plane at *B*. Therefore *A–B* is the intersection of the horizontal projecting plane of *Q–R* and the plane of the two given lines *M–N* and *O–P*. *Q–R* crosses *A–B* at *D*, which by analysis is the required point of intersection.

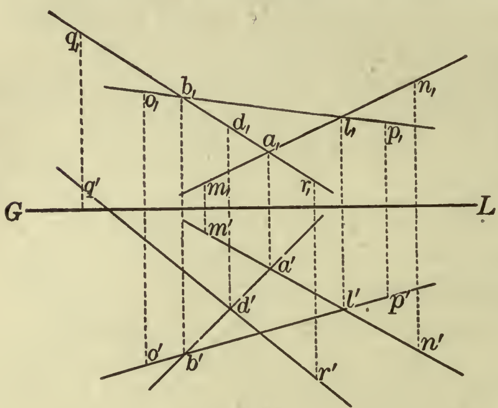

FIG. 68

Check. Use the vertical projecting plane of *Q–R*.

152. Problem 90. *Find the point in which the plane determined by the two lines* [*M* = – 6, 6, 6; *N* = 0, 2, 1] *and* [*M* = – 6, 6, 6; *O* = 2, 8, 4] *is intersected by the line* [*Q* = – 4, 2, 2; *R* = 2, 6, 8].

153. Problem 91. *Find the point in which the plane determined by the two lines* [*M* = – 4, – 2, – 8; *N* = – 4, – 8, – 2] *and* [*M* = – 4, – 2, – 8; *O* = 4, – 7, – 3] *is intersected by the line* [*Q* = – 6, – 6, – 2; *R* = 6, – 6, – 8].

154. Problem 92. *To draw through a given point a straight line perpendicular to a given plane, and to determine the distance of the point from the plane.*

Analysis. By Section 44 we know that the horizontal projection of the required line must be perpendicular to the horizontal trace of the given plane, and that the vertical projection of the required line must be perpendicular to the vertical trace.

The distance of a point from a plane is measured on a straight line drawn from the point perpendicular to the plane. If then we find the point in which such a line intersects the given plane, the distance from the given point to this point of intersection will be the required distance.

Construction. See Fig. 69. Let S represent the given plane and let O represent the given point. Through $o_{\prime}$ draw $o_{\prime}-d_{\prime}$ perpendicular to $S-s_{\prime}$ and through o' draw $o'-d'$ perpendicular to $S-s'$. $O-D$ is the required line.

By Problem 89 find the point D in which the line $O-D$ intersects S. The distance $O-D$, which is found by revolving the horizontal projecting plane of $O-D$ about $o_{\prime}-d_{\prime}$ into H (see Problem 72), is the required distance.

Check. To test the perpendicularity of the line, draw through D some line of the plane and note whether this line is perpendicular to $O-D$. To test the distance of the point from the plane, revolve the vertical projecting plane of $O-D$ about $o'-d'$ into V.

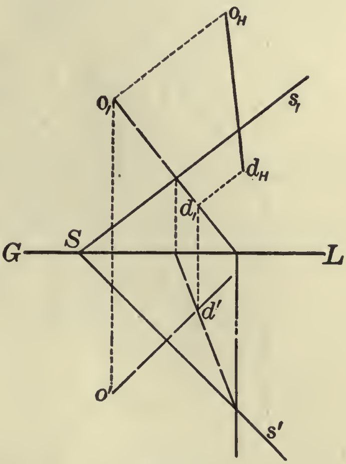

Fig. 69

155. Problem 93. *Find the distance of the point* $M = -2, -5, -8$ *from the plane* $S = 0°, \propto, 2.$

156. Problem 94. *Find the distance of the point* $M = -2, -6, -6$ *from the plane* $S = 0°, 2, \propto.$

157. Problem 95. *Find the distance of the point* $M = 0, -6, -6$ *from the plane* $S = -4, 300°, 270°.$

158. Problem 96. *Find the distance of the point* $M = 0, -6, -6$ *from the plane* $S = -4, 270°, 330°.$

159. Problem 97. *Find the distance of the point* $M = 0, -8, 6$ *from the plane* $S = 0, 210°, 60°.$

160. Problem 98. *Find the distance of the point* $M = 0, -6, -6$ *from the plane* $S = 0°, -3, -6.$

161. Problem 99. *To project a given straight line upon a given plane.*

Principle. The projection of a straight line upon a plane is the line in which a plane drawn through the given line perpendicular to the given plane intersects the given plane (see Section 23).

Analysis 1. If through any point of the given line we draw a straight line perpendicular to the given plane, the plane of the given line and this auxiliary line will be the projecting plane of the given line, and will intersect the given plane in the required projection.

Analysis 2. Since the projection of a straight line upon a plane is a straight line and contains the projections of all the points of the given line, the required projection is the straight line determined by the projections of any two points of the given line.

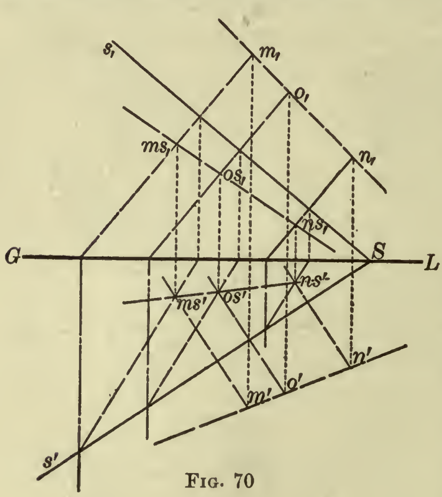

FIG. 70

Construction. See Fig. 70. Let $M–N$ represent the given line and let S represent the given plane. Following Analysis 2, project upon the given plane any two points of the given line, as M and N. The projection of M upon S is MS, horizontally projected at $ms_,$ and vertically projected at ms' (see Sections 5 and 151).

The projection of N upon S is NS, horizontally projected at $ns_,$ and vertically projected at ns'. Therefore $ms_,–ns_,$ is the horizontal projection and $ms'–ns'$ is the vertical projection of the required projection.

Check. Assume a third point O upon the given line. Project this point upon S, and note if this projection falls upon the projection already found.

162. Problem 100. *Project the line* [$M = -2, -3, -8$; $N = 3, -9, -4$] *upon the plane* $S = -8$, 315°, 330°, *following Analysis 1.*

163. Problem 101. *Project the line* [$M = -6, 6, 8$; $N = 6, -3, 6$] *upon the plane* $S = 0°, -3, 6$.

164. Problem 102. *Project the line* $[M = -4, -4, -4;\ N = 0, -8, -6]$ *upon the plane* $S = -6, 315°, 330°$, *following Analysis 2.*

165. Problem 103. *To pass a plane through a given point and perpendicular to a given straight line.*

Analysis. From Section 44 we know that the horizontal and vertical traces of the required plane will be perpendicular respectively to the horizontal and vertical projections of the given line. We know, then, the direction of each of the required traces. If a straight line be drawn through the given point parallel to either of these traces, it must be a line of the required plane, and, unless parallel to $G\text{–}L$, will intersect one of the planes of projection in a point of the corresponding trace of the required plane. A straight line through the point thus found and perpendicular to the corresponding projection of the given line will be one of the required traces.

The other required trace will pass through the point in which the trace just found intersects $G\text{–}L$, and will be perpendicular to the remaining projection of the given line.

Construction. See Fig. 71. Let $M\text{–}N$ represent the given line, and let O represent the given point. Through O draw $O\text{–}A$ parallel to the horizontal trace of the required plane.

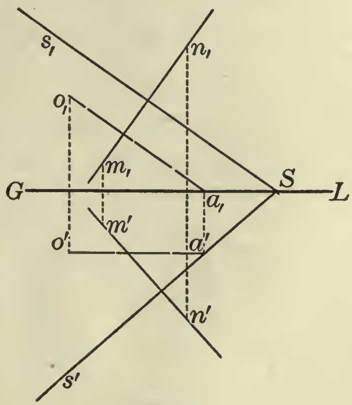

FIG. 71

Since this horizontal trace is to be perpendicular to $m_{\prime}\text{–}n_{\prime}$, the horizontal projection of $O\text{–}A$ will be perpendicular to $m_{\prime}\text{–}n_{\prime}$. Since $O\text{–}A$ is parallel to a line in H, its vertical projection will be parallel to $G\text{–}L$.

The point a' in which $O\text{–}A$ intersects V is a point in the vertical trace of the required plane, and $S\text{–}s'$ drawn through a' perpendicular to $m'\text{–}n'$ is the required vertical trace. $S\text{–}s_{\prime}$ drawn through S perpendicular to $m_{\prime}\text{–}n_{\prime}$ is the required horizontal trace.

Check. Draw a line through O parallel to the required vertical trace, and note whether such a line intersects H in the horizontal trace already found.

166. Problem 104. *Draw a plane through the point* $O = -2, 4, 2$ *and perpendicular to the line* [$M = -6, -6, -6$; $N = 6, -3, -1$].

167. Problem 105. *Draw a plane through the point* $O = 2, -6, -2$ *and perpendicular to the line* [$M = -2, 1, 6$; $N = -2, 6, 1$].

168. Problem 106. *Draw a plane through the point* $O = -4, 9, 4$ *and perpendicular to the line* [$M = -6, -4, 4$; $N = 6, 3, -3$].

169. Problem 107. *To pass a plane through a given point and parallel to two given straight lines.*

Analysis. If through the given point we draw two straight lines parallel respectively to the two given lines, the plane of these two lines will be parallel to the two given lines, since a plane is parallel to a straight line when it contains a straight line parallel to that line.

Construction. See Fig. 72. Let M–N and O–P represent the two given lines, and let Q represent the given point. Through Q draw A–B parallel to M–N; also through Q draw C–D parallel to O–P. A–B intersects H at $b_{,}$ and intersects V at a'. C–D intersects H at $c_{,}$ and intersects V at d'. Therefore S–$s_{,}$, which is drawn through $b_{,}$ and $c_{,}$, is the horizontal trace of the required plane, and S–s', which is drawn through a' and d', is the vertical trace of the required plane.

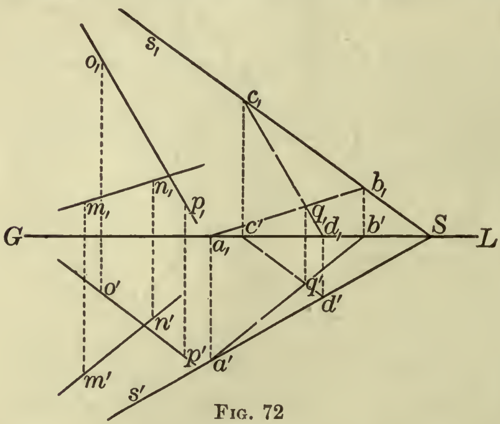

Fig. 72

Check. By Section 43 assume any point in the plane now determined. Through this point draw two straight lines, one parallel to one of the given lines and the other parallel to the other. The points in which these two lines intersect H and V should lie in the horizontal and vertical traces already located.

170. Problem 108. *Pass a plane through the point* $Q = -4, -4, -2$ *and parallel to the two lines* [$M = 0, 2, 6$; $N = 8, -3, 1$] *and* [$O = 4, 4, 1$; $P = 4, -2, 6$].

171. Problem 109. *Pass a plane through the point* $Q = -2, -6, 3$ *and parallel to the two lines* $[M = -4, 5, -3;\ N = 4, -3, 6]$ *and* $[O = -2, -1, 5;\ P = 6, -1, 5]$.

172. Problem 110. *To pass a plane through a given point and parallel to a given plane.*

Analysis. The traces of the required plane will be parallel to the corresponding traces of the given plane, and will be fully known when one point in each trace is determined. A straight line through the given point and parallel to either trace of the given plane will be a line of the required plane and will intersect H or V in the corresponding trace of the required plane.

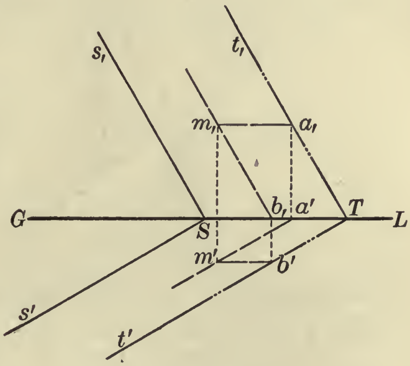

FIG. 73

Construction. In Fig. 73 let S represent the given plane and let M represent the given point.

Through M draw M–A parallel to S–s' and produce it to meet H at $a_{\prime}$. The point $a_{\prime}$ is in the horizontal trace of the required plane.

Through M draw M–B parallel to S–$s_{\prime}$ and produce it to meet V at b'. The point b' is in the vertical trace of the required plane.

The two required traces, T–$t_{\prime}$ and T–t', the former parallel to S–$s_{\prime}$ and the latter parallel to S–s', are now located.

Check. The two traces now determined should meet on G–L.

173. Problem 111. *Pass a plane through the point* $M = -6, -4, -6$ *and parallel to the plane* $S = -6, 40°, 60°$.

174. Problem 112. *Pass a plane through the point* $M = 0, 3, 6$ *and parallel to the plane* $S = 0°, 5, 3$.

175. Problem 113. *To pass a plane through a given straight line and parallel to another given straight line.*

Analysis. Through any point of the first line draw a straight line parallel to the second line. The plane of the first line and the auxiliary line will be the required plane.

Construction. See Fig. 74. Let M–N represent the line through which the plane is to be passed, and let O–P represent the line to which the plane is to be parallel.

Through any point of M–N, as M, draw M–Q parallel to O–P and produce it to intersect H in $c_{,}$ and to intersect V in d'. Find the point $a_{,}$ in which M–N intersects H; also find the point b' in which M–N intersects V. Through $c_{,}$ and $a_{,}$ draw S–$s_{,}$, and through b' and d' draw S–s'. These are the traces of the required plane and should cross G–L at the same point.

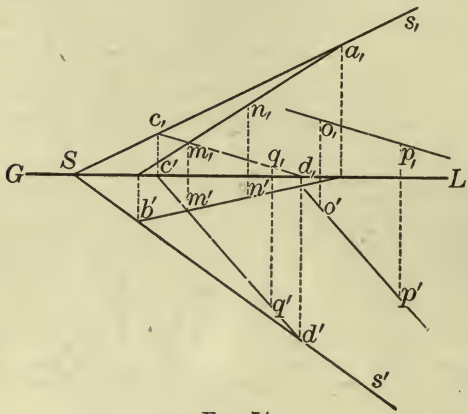

Fig. 74

Check. Through some other point of M–N draw another straight line parallel to O–P and note whether this line intersects H and V in the traces now located.

176. Problem 114. *Pass a plane through* [$M = -6, 2, 6$; $N = 4, -5, 1$] *and parallel to* [$O = 1, -1, 6$; $P = 1, 5, 2$].

177. Problem 115. *Pass a plane through* [$M = 5, -6, -3$; $N = -4, 1, 8$] *and parallel to* [$O = -5, 3, 5$; $P = 2, 3, 5$].

178. Problem 116. *Pass a plane through* [$M = -6, 4, -6$; $N = 5, 4, -6$] *and parallel to* [$O = 2, 2, 6$; $P = 2, -6, 1$].

179. Problem 117. *To find the shortest distance from a given point to a given straight line.*

The required distance must be measured upon a straight line drawn from the given point perpendicular to the given line.

Analysis 1. If we pass a plane through the given point and the given line, and revolve this plane about one of its traces into the corresponding plane of projection, the line and the point will be shown in their true relation. From the revolved position of the point draw a straight line perpendicular to the revolved position of the line, and upon this line measure the required distance.

Analysis 2. Draw through the given point a plane perpendicular to the given line and find the point in which this plane is

intersected by the given line. A straight line from this point of intersection to the given point will be perpendicular to the given line and will therefore be the line upon which the required distance may be measured.

Construction. See Fig. 75. Let M–N represent the given line and let O represent the given point.

Following Analysis 1, draw the line O–P through O parallel to M–N and produce it to intersect H at $p_{\prime}$. Find $a_{\prime}$, the point in which M–N intersects H. S–$s_{\prime}$, drawn through $p_{\prime}$ and $a_{\prime}$, is the horizontal trace of a plane containing M–N and O.

Revolve this plane about S–$s_{\prime}$ as an axis, into H. O will fall at o_H, M will fall at m_H, and $a_{\prime}$ will remain stationary. Since A and M are points of the line M–N, $a_{\prime}$–m_H is the revolved position of the given line.

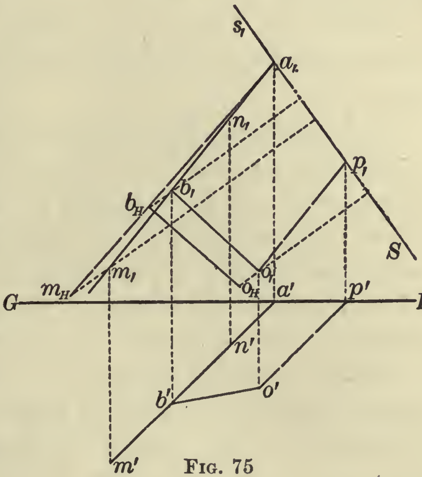

FIG. 75

Through o_H draw o_H–b_H perpendicular to $a_{\prime}$–m_H. The distance o_H–b_H is the required distance.

Check. Revolve the plane of the point and the line about its vertical trace into V, or solve according to Analysis 2.

180. Problem 118. *Find the distance from the point* $O = 2, 4, 6$ *to the line* $[M = -4, -6, 4;\ N = 4, -6, -4]$.

181. Problem 119. *Find the distance from the point* $O = 4, 6, 2$ *to the line* $[M = -2, 2, 6;\ N = -2, -6, -5]$.

182. Problem 120. *To draw through a given point a straight line perpendicular to a given straight line.*

Analysis. This problem is embodied in Problem 117, since in finding the shortest distance from a point to a straight line, a straight line is drawn from the given point perpendicular to the given line.

If in the solution of Problem 117, Analysis 2 is followed, the projections of the line required in Problem 120 will be found directly.

If in the solution of Problem 117, Analysis 1 is followed, a counter revolution will be necessary in order to determine the projections of the line required in Problem 120.

Construction. Making use of the results obtained in Fig. 75, revolve the plane S back to its original position. O will take the position $(o_{\prime}, o')$ which it originally occupied, and B, which is a point on M–N, will move back in a plane perpendicular to S–$s_{\prime}$ until its horizontal projection takes the position $b_{\prime}$ and its vertical projection takes the position b' in the straight line through $b_{\prime}$ perpendicular to G–L.

The lines $o_{\prime}$–$b_{\prime}$ and o'–b' represent respectively the horizontal and vertical projections of the required line through O perpendicular to M–N.

Check. Follow Analysis 2, Problem 117.

183. Problem 121. *Draw through the point* $O = 0, 4, -2$ *a straight line perpendicular to the line* $[M = 0,\ 2,\ 6\,;\ N = 0,\ -5,\ -2]$.

184. Problem 122. *Draw through the point* $O = 4, -2, 5$ *a straight line perpendicular to the line* $[M = -4,\ 2,\ -5\,;\ N = -4,\ -3,\ 6]$.

185. Problem 123. *To find the angle which a given straight line makes with a given plane.*

The angle which a straight line makes with a plane is the angle which the line makes with its projection on that plane.

Analysis 1. By Problem 99 project the given line upon the given plane, and by Problem 80 measure the angle between the line itself and its projection.

Analysis 2. If through any point of the given line a straight line be drawn perpendicular to the given plane, it will intersect the given plane in one point of the projection of the given line upon this plane. The line itself will intersect the plane in another point of this projection. A straight line through the two points just found will be the projection of the line upon the plane.

The line itself, the projection of the line, and the straight line drawn perpendicular to the plane, together, form a right-angled triangle. In this triangle the line itself is the hypotenuse, the projection of the line is the base, and the straight line perpendicular to the plane is the altitude.

The oblique angle at the base of the triangle is the required angle, and since the triangle is right-angled the angle at the vertex must be the complement of the required angle.

Construction. See Fig. 76. Let S represent the given plane and let $M\text{–}N$ represent the given line.

Following Analysis 2, through some point of $M\text{–}N$, as M, draw $M\text{–}B$ perpendicular to S and produce it to intersect H at $b_{\prime}$. Find the point $a_{\prime}$ in which $M\text{–}N$ intersects H. The line $a_{\prime}\text{–}b_{\prime}$ is the horizontal trace of the plane of the two lines $M\text{–}N$ and $M\text{–}B$.

Revolve this plane about $a_{\prime}\text{–}b_{\prime}$ as an axis, into H. The points $a_{\prime}$ and $b_{\prime}$ remain stationary and M revolves to m_H. The angle $a_{\prime}\text{–}m_H\text{–}b_{\prime}$ is the true size of the angle between the given line and the line perpendicular to the plane, and is, by analysis, the complement of the required angle.

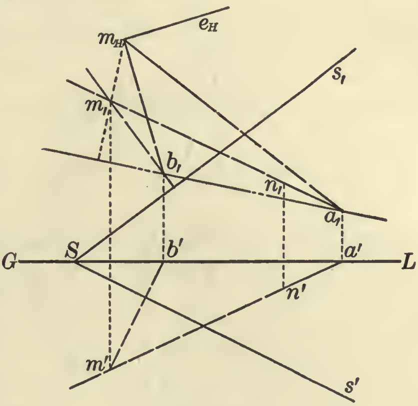

FIG. 76

Draw $m_H\text{–}e_H$ perpendicular to $m_H\text{–}b_{\prime}$. The angle $a_{\prime}\text{–}m_H\text{–}e_H$, which by construction is the complement of the angle $a_{\prime}\text{–}m_H\text{–}b_{\prime}$, is the required angle.

Check. Solve by Analysis 1, or assume some point other than M, in $M\text{–}N$, as the point through which to draw the line perpendicular to the plane.

186. Problem 124. *Find the angle between the line* $[M = -6, 3, 5;$ $N = 5, -4, 2]$ *and the plane* $S = 0°, 4, 6$.

187. Problem 125. *Find the angle between the line* $[M = -6, 3, 5;$ $N = 5, -4, 1]$ *and the plane* $S = 0, 90°, 90°$.

188. Problem 126. *Find the angle between the line* $[M = 0, 6, -1;$ $N = 0, -5, -6]$ *and the plane* $S = 0°, 6, 3$.

189. Problem 127. *To find the angle between two given planes.*

A dihedral angle is measured by the plane angle formed by two straight lines perpendicular to the edge at the same point, one line lying in one face and the other line lying in the other face.

Analysis 1. Pass a plane perpendicular to the line of intersection of the two given planes. This plane will intersect the given planes in straight lines perpendicular to the line of intersection at the same point. The angle between these two lines is the required angle.

Analysis 2. Assume any point between the faces of the dihedral angle. Through this point draw two straight lines, one perpendicular to one face and the other perpendicular to the other face. The angle formed by these two lines will be the supplement of the required angle.

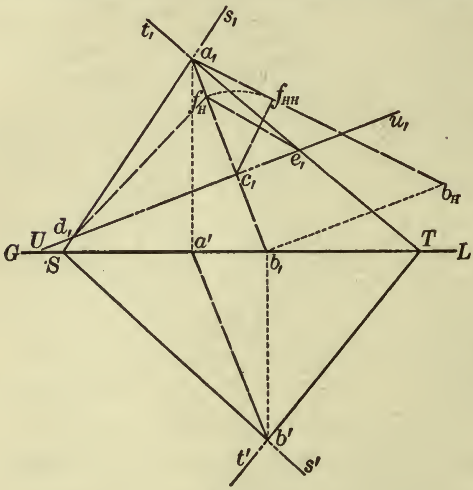

Fig. 77

Construction. See Fig. 77. Let S and T represent the given planes which intersect in the line A–B.

Following Analysis 1, through any point of a_1–b_1, as c_1, draw U–u_1 perpendicular to a_1–b_1. U–u_1 may be taken as the horizontal trace of a plane U perpendicular to A–B (see Section 45). The line cut from S by U, — the line forming one side of the required plane angle, — pierces H at d_1; the line cut from T by U, — the line forming the other side of the same angle, — pierces H at e_1, and it only remains to find the vertex of the angle, which is the point in which U is intersected by A–B.

U will cut the horizontal projecting plane of A–B in a straight line perpendicular to A–B, piercing H at c_1, and intersecting A–B at the vertex of the angle sought.

Revolve the horizontal projecting plane of A–B about a_1–b_1 as an axis, into H. The line a_1–b_H is the revolved position of A–B, and c_1–f_{HH} drawn through c_1 perpendicular to a_1–b_H is the revolved position of the intersection of U and the horizontal projecting plane of A–B. The point f_{HH} is the revolved position of the vertex, and

the distance $c_{\prime}$–f_{HH} is the distance of the vertex, in true position, from the line U–$u_{\prime}$.

With the vertex in true position revolve U about U–$u_{\prime}$ as an axis, into H. The points $d_{\prime}$ and $e_{\prime}$ will remain stationary and the vertex F will fall at f_H, where $c_{\prime}$–f_H is equal to $c_{\prime}$–f_{HH}.

The angle $d_{\prime}$–f_H–$e_{\prime}$ is the measure of the required angle.

Check. Pass another plane perpendicular to A–B.

190. Problem 128. *Find the angle between the two planes* $S = -5, 140°, 40°$ *and* $T = 5, 30°, 120°$.

191. Problem 129. *Find the angle between the two planes* $S = -2, 90°, 90°$ *and* $T = 4, 30°, 60°$.

192. Problem 130. *Find the angle between the two planes* $S = 0°, -6, -2$ *and* $T = 0°, -2, -5$.

193. Problem 131. *Find the angle between the plane* $S = -4, 30°, 60°$ *and the plane* H.

Suggestion. In this case the edge of the dihedral angle is the horizontal trace of the given plane. The auxiliary plane, which is perpendicular to the edge, is therefore perpendicular to H. Its horizontal trace will be perpendicular to the horizontal trace of the given plane, and its vertical trace will be perpendicular to G–L.

194. Problem 132. *Find the angle between the plane* $S = -3, 20°, 60°$, *and the plane* V.

195. Problem 133. *Given either trace of a plane and the angle which the plane makes with the corresponding plane of projection; required to determine the other trace.*

Analysis. Suppose the horizontal trace of the plane and the angle which the plane makes with H are given.

Any plane perpendicular to this horizontal trace will cut from the plane whose vertical trace is sought a straight line, and from H another straight line, both of which will be perpendicular to the horizontal trace at the same point. These two lines will form a plane angle measuring the given dihedral angle, and if revolved into H about the side in H, will be seen in true size.

Through any point of the given horizontal trace draw two straight lines in H, the first perpendicular to the trace and the second making an angle with the first equal to the measure of the given dihedral angle.

Revolve the plane of these two lines about the first line as an axis until perpendicular to *H*. The second line in this position must be a line of the plane whose vertical trace is sought, and must pierce *V* at a point in this trace.

By assuming other points in the given horizontal trace and proceeding as above, other points in the required vertical trace may be obtained.

Construction. See Fig. 78. Let S–$s_{,}$ represent the given horizontal trace, and let the angle A represent the measure of the given angle which the plane makes with *H*.

Through $b_{,}$, any point on S–$s_{,}$, draw $b_{,}$–$d_{,}$ perpendicular to S–$s_{,}$. Through $b_{,}$ also draw $b_{,}$–d_{II}, making the given angle A with $b_{,}$–$d_{,}$. The line $b_{,}$–d_{II} is, by analysis, the revolved position of a line of the plane whose vertical trace is sought.

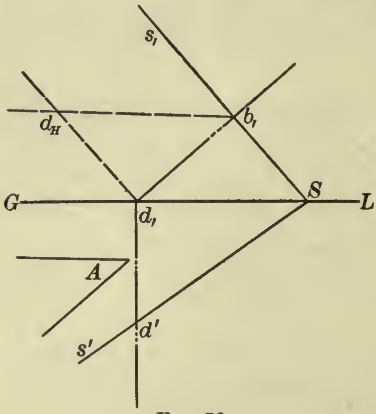

Fig. 78

Through $d_{,}$ draw $d_{,}$–d' perpendicular to G–L; also through $d_{,}$ draw $d_{,}$–d_{II} perpendicular to $b_{,}$–$d_{,}$, intersecting $b_{,}$–d_{II} at d_{II}.

Revolve the plane of the three lines, $b_{,}$–$d_{,}$, $b_{,}$–d_{II}, and $d_{,}$–d_{II} about $b_{,}$–$d_{,}$ as an axis until the plane occupies its true position, which is perpendicular to *H*. The line $d_{,}$–d_{II} will take the position $d_{,}$–d' perpendicular to G–L, and the point d_{II} will take the position d' on the line $d_{,}$–d', and at the distance $d_{,}$–d_{II} below G–L. The point d', then, is the point in which B–D, in true position, pierces *V*, and is therefore a point in the required vertical trace. S–d'–s' is the required vertical trace.

In case the given horizontal trace does not intersect G–L within the limits of the drawing, assume another point upon the horizontal trace and proceed as above to find another point in the vertical trace.

Check. Assume some point on the horizontal trace other than those already used, and proceed as above to locate an additional point on the vertical trace already determined.

196. Problem 134. *Given the horizontal trace* [$S = -4, 3, 0$; $s_{\prime} = 4, 6, 0$] *of a plane S, and given the angle* 60 *degrees which the plane S makes with H; required the vertical trace of the plane S.*

197. Problem 135. *Given the vertical trace* [$S = -5, 0, 0$; $s' = 3, 0, 5$] *of the plane S, and given the angle* 30 *degrees which the plane S makes with V; required the horizontal trace of the plane S.*

198. Problem 136. *Given the horizontal trace* [$S = -5, -4, 0$; $s_{\prime} = 5, -4, 0$] *of the plane S, and given the angle* 60 *degrees which the plane S makes with H; required the vertical trace of the plane S.*

199. Problem 137. *Given two straight lines not in the same plane; required to draw a third straight line which shall be perpendicular to both.*

Analysis 1. Through one of the lines pass a plane parallel to the other line, and project the second line upon this plane. This projection must be parallel to the second line and intersect the first line. At this point of intersection erect a straight line perpendicular to the plane.

This line must be perpendicular both to the first line and to the projection of the second line. It will remain in the projecting plane of the second line and therefore intersect the second line. It will be perpendicular to the second line since it is perpendicular to the projection of the second line on a plane to which the second line is parallel. It is therefore perpendicular both to the first line and to the second line.

Analysis 2. In case we are required to find simply the shortest distance between two straight lines not in the same plane, we may proceed as follows: Through any point of one of the lines pass a plane perpendicular to the other line. Project the first line upon this plane. The distance from the point in which the second line intersects the plane to the projection of the first line upon the plane is the required distance.

Construction. See Fig. 79. Let *M–N* and *O–P* represent the given lines.

Following Analysis 1, draw through *M–N* the plane *S* parallel to *O–P*. To do this, draw through any point of *M–N*, as *D*, the line *D–E* parallel to *O–P*. Find the point $e_{\prime}$ in which *D–E* pierces

H; also find the point $a_{,}$ in which M–N pierces H, thus locating the horizontal trace S–$s_{,}$. Find the point b' in which M–N pierces V; also find the point f' in which D–E pierces V, thus locating the vertical trace S–s'.

Project O–P upon S. To do this, draw through any point of O–P, as K, the line K–KS perpendicular to S, and find its intersection, KS, with S. The point KS is one point in the projection of O–P upon S (see Section 161), and since O–P is parallel to S, KS–X drawn parallel to O–P is the required projection.

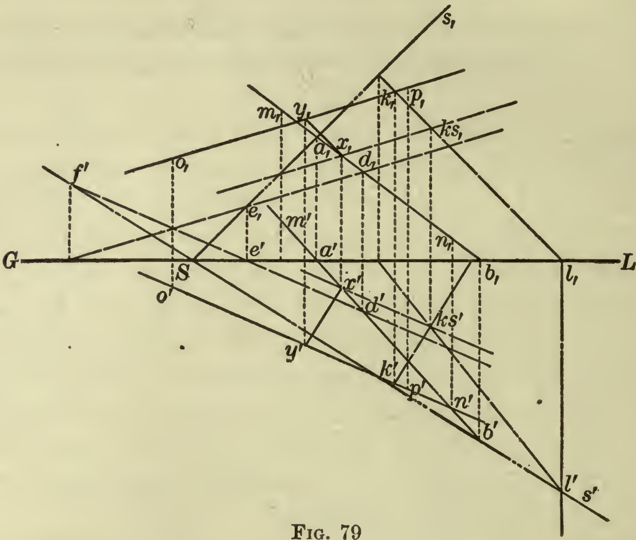

FIG. 79

At the point X, where KS–X crosses M–N, draw X–Y perpendicular to S (see Section 154), and note the point Y in which it intersects O–P. $X-Y$ is the required line, and if the true length of its intercept between M–N and O–P is desired, it may be found by Problem 72.

Check. The points $x_{,}$ and x', which are found independently, should fall in the same straight line perpendicular to G–L. The same should be true of $y_{,}$ and y'.

A more severe check would be made by drawing the auxiliary plane, in the first part of the construction, through O–P parallel to M–N instead of through M–N parallel to O–P.

In this way the problem is solved by a method which in theory is the same as that explained above, but which in application is entirely independent of it.

In the above problem, if one of the two given lines is perpendicular to the plane of projection, the required line will be parallel to the plane of projection, and its own projection will pass through the point projection of the line which is perpendicular to the plane and will be perpendicular to the projection of the other given line.

If one of the two given lines is parallel to the plane of projection, the projection of the required line upon this plane will be perpendicular to the projection of the parallel line upon this plane.

200. Problem 138. *Given the two straight lines* $[M = -5, 3, 5;$ $N = 5, 3, 5]$ *and* $[O = -4, 2, 6;$ $P = 4, -4, -3]$; *required the projections of the straight line perpendicular to both.*

201. Problem 139. *Given the two straight lines* $[M = -5, 3, 5;$ $N = 5, 3, 5]$ *and* $[O = 2, -6, 7;$ $P = 2, 1, -6]$; *required the projections of the straight line perpendicular to both.*

202. Problem 140. *Given the two straight lines* $[M = -5, -6,$ $-6;$ $N = 3, 1, -2]$ *and* $[O = -4, -6, -3;$ $P = 4, -7, -6]$; *required the shortest distance between the two lines.*

CHAPTER VII

GENERATION AND CLASSIFICATION OF LINES

203. Generation of Lines. We may regard a line as the generation resulting from the movement of a point which in the course of the generation occupies an infinite number of consecutive positions at infinitely small distances apart.

The portion of the line generated by the point while moving from one position to its consecutive position is called an *elementary line*, and while in theory it may be regarded as having length, practically speaking it has none.

204. Classification of Lines. If the generating point moves continuously in one direction, the line so generated is called a *straight* or a *right line.*

If the generating point is constantly changing its direction of movement, the line so generated is called a *curved line*, or a *curve.*

When the generating point of a curved line confines its movement to a plane the line is called a *curve of single curvature.*

When the generating point of a curved line does not confine its movement to a plane, the line is called a *curve of double curvature.*

205. Curves of Single and of Double Curvature. The character of a curve, whether of single or of double curvature, will depend upon the law which governs the motion of the generating point, and there may be as many distinct curves as there are distinct laws governing the motion of a point.

If the generating point moves in a plane and retains a constant distance from a fixed point in the plane, it will generate a curve of single curvature called the *circle.*

If the generating point moves in a plane and in such a way that the sum of its distances from two fixed points in the plane is a constant quantity, it will generate a curve of single curvature called the *ellipse.*

If the generating point moves in a plane, and in such a way that its distance from a fixed point in the plane is equal to its distance from a fixed straight line in the plane, it will generate a curve of single curvature called the *parabola.*

If the generating point moves in a plane, and in such a way that the difference of its distances from two fixed points in the plane is a constant quantity, it will generate a curve of single curvature called the *hyperbola.*

If the generating point moves in such a way as to retain a constant distance from a fixed straight line, and to have a uniform motion both around and in the direction of the straight line, it will generate a curve of double curvature called the *helix.*

206. Representation of Curves. Curved lines like straight lines may be represented by their projections on H and V, and when so represented they are in general definitely determined.

These projections are the loci of the projections of the generating point in its consecutive positions, and are usually curved lines.

If the curve is of single curvature and its plane is parallel to the plane of projection, its projection upon this plane is a curve of the same character and magnitude as the original curve.

If the curve is of single curvature and its plane is perpendicular to the plane of projection, its projection upon this plane is a straight line.

The projection of a circle upon a plane to which the plane of the circle is oblique is an ellipse.*

The projection of an ellipse upon a plane to which the plane of the ellipse is oblique is either a circle or an ellipse.*

The projection of a curve of double curvature is always a curve whatever the relation of the curve to the plane of projection. The projection of the helix upon a plane perpendicular to its axis is a circle.

207. Tangents to Curves. A straight line is tangent to a curve at a given point when it represents the rectilinear path in which the generating point is moving at the instant it passes through the point of tangency.

*The truth of these statements will be more apparent after the student has obtained some knowledge of the nature of the intersections of cylindrical surfaces by planes. See Chapter XVI.

In Fig. 80 let A–B–D represent a curve generated by a point moving under some law. Suppose that when the generating point reaches the position B, the law under which it is moving ceases to act and the point is allowed to move freely in the direction B–E, in which it is moving at this instant. The straight line B–E is tangent to the curve at the point B.

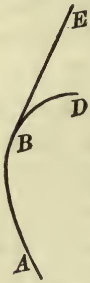

Fig. 80

Or, a straight line is tangent to a curve at a given point when the line contains the given point and its consecutive point. In Fig. 81 let B represent any point on the curve A–B–D. Through B draw any secant, as B–F, cutting the curve at B and F. Now if the point B remain fixed and the point F be made to move along the curve toward B, the secant will gradually approach the position of a straight line tangent to the curve at the point B. Finally, when the point F has taken the position consecutive to B, or practically the position B itself, the line B–F will take the position B–E and be tangent to the curve at the point B.

Two curves are tangent to each other when they have two consecutive points in common, or when they are both tangent to the same line at a common point.

A straight line which is tangent to a curve of single curvature lies in the plane of the curve.

If two lines, straight or curved, are tangent to each other, their projections will also be tangent to each other, since the projections of the two consecutive points common to the two lines will also be consecutive and be common to the two projections of these lines.

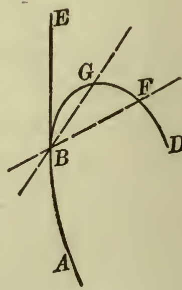

Fig. 81

208. Problem 141. *To draw a rectilinear tangent to an ellipse at a point assumed on the curve.**

Construction.† In Fig. 82 let A–B and D–E represent the axes of the ellipse, let F and F' represent the foci of the ellipse, and let P represent the point assumed on the curve.

* The method of representing the ellipse, the parabola, and the hyperbola is fully explained in text-books on elementary mechanical drawing.

† The proof of the methods here given for drawing rectilinear tangents to the ellipse, the parabola, and the hyperbola may be found in text-books on analytic geometry.

Draw the focal radii F–P and F'–P, and bisect their external angle by the line P–O. The line P–O is the required tangent.

209. Problem 142. *To draw a rectilinear tangent to a parabola at a point assumed on the curve.*

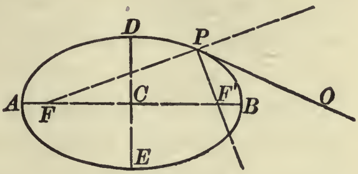

FIG. 82

Construction. In Fig. 83 let A–C represent the axis of the parabola, let A–B represent the directrix, let D represent the vertex, and let P represent the point assumed on the curve.

Draw P–L perpendicular to A–C. Lay off D–O equal to D–L. The straight line O–P through O and P is the required tangent.

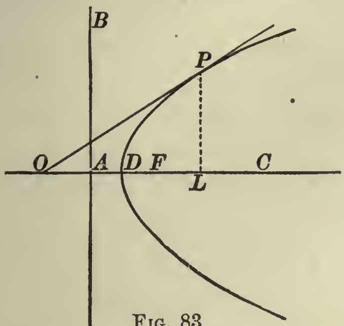

FIG. 83

210. Problem 143. *To draw a rectilinear tangent to a hyperbola at a point assumed on the curve.*

Construction. In Fig. 84 let A–B represent the axis of the hyperbola, let F and F' represent the foci, let A and B represent the vertices of the two branches of the curve, and let P represent the point assumed on the left-hand branch of the curve.

Draw the focal radii P–F and P–F' and bisect the angle F–P–F' by the line P–O. The line P–O is the required tangent.

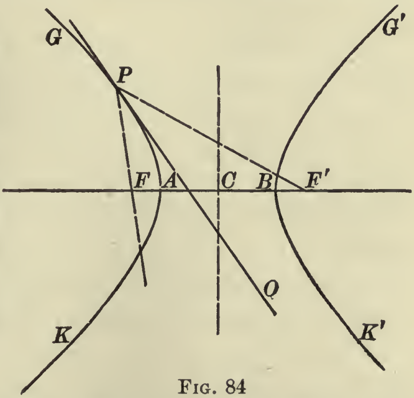

FIG. 84

211. Normals to Curves. A straight line is normal to a curve at a given point when it is perpendicular to the rectilinear tangent to the curve at that point.

A normal to a curve of single curvature, like the rectilinear tangent, lies in the plane of the curve.

212. Rectification of Curves. A curve is said to be rectified when its length in linear units is determined.

If a curve is rolled out upon a rectilinear tangent to the curve, so that the consecutive points of the curve fall consecutively upon the tangent, that portion of the tangent covered between the first and last points of contact will represent the rectification of that portion of the curve between these same points.

The work of rectification is accomplished graphically by dividing the curve into a large number of small arcs, so small that the chords of the arcs may for practical purposes be taken as equal in length to the arcs themselves, and by taking the summation of these chords.

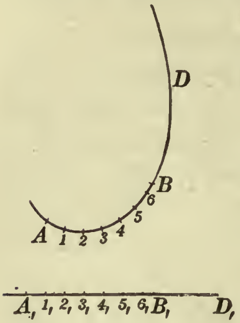

Fig. 85

In Fig. 85 let it be required to rectify the portion A–B of the curve A–B–D. Set the dividers at some sufficiently small distance and apply this distance as a chord successively to the curve, starting with the point A. In this particular case the chord is applied six times, leaving the little arc 6–B whose chord is smaller than the chord assumed.

Now apply the original chord distance six times to the straight line $A_{,}$–$D_{,}$, starting with $A_{,}$, adding to the sum the little distance $6_{,}$–$B_{,}$, which is the chord measure of the little arc 6–B on the original curve. The distance $A_{,}$–$B_{,}$ is the rectified length of the arc A–B.

213. The Helix. The helix is a curve of double curvature generated by a point moving uniformly both around and in the direction of a given straight line from which it retains a constant distance.

The straight line around and along which the generating point moves is called the *axis* of the helix.

The constant distance which the generating point retains from the axis is called the *radius* of the helix.

The distance over which the generating point moves in the direction of the axis during one circuit of the axis is called the *pitch* of the helix.

If, to an observer looking along the axis in that direction in which the generating point is moving, the circuit of the generating point about the axis is clockwise, the helix is called a *right-handed helix.*

If, under the same conditions, the circuit is contra-clockwise, the helix is called a *left-handed helix*.

A helix is completely known when the radius, the pitch, and the direction of the circuit (whether right-handed or left-handed) are given.

214. To represent the Helix. See Fig. 86. Let A–B represent the axis of the helix, assumed for convenience perpendicular to H.

With $a_{\prime}$ as a center and with $a_{\prime}$–$d_{\prime}$ as a radius, equal in length to the radius of the helix, draw the circle $d_{\prime}$–$d_{\prime\prime}$–$d_{\prime\prime\prime}$–$\cdots$–d_{IX}. The circumference of this circle will be the horizontal projection of the helix, since the generating point retains a constant radial distance from the axis.

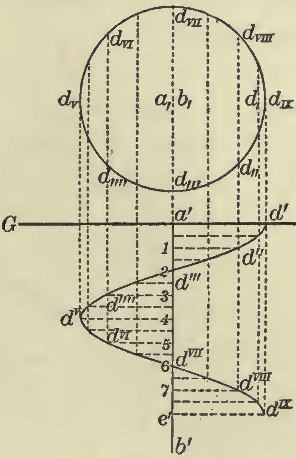

Fig. 86

Let D ($d_{\prime}$, d') represent the position of the generating point when in H. Beginning with $d_{\prime}$ divide the circumference of the circle into any number, say eight, equal parts. The points of division are $d_{\prime}$, $d_{\prime\prime}$, $d_{\prime\prime\prime}$, $\cdots$, d_{IX}. Lay off upon a'–b', downward from G–L, a distance a'–e' equal to the pitch of the helix. Divide this distance a'–e' into the same number of equal parts as the circumference of the circle has been divided into, and through the points of division, 1, 2, 3, 4, $\cdots$, e', draw horizontal straight lines.

While the generating point is making one circuit of the axis it moves through a vertical distance equal to a'–e'. Therefore, since the two motions are uniform, while the generating point is making any fractional part of the circuit it must move through the same fractional part of the total vertical distance.

When the generating point is on H its horizontal projection is at $d_{\prime}$ and its vertical projection is at d'.

When the generating point has made one eighth of its circuit its horizontal projection will be at $d_{\prime\prime}$. Its vertical projection must be on a straight line through $d_{\prime\prime}$ perpendicular to G–L, and on a straight line through 1 parallel to G–L, and therefore at d$''$.

If we draw vertical lines through $d_{,,,}$, $d_{,,,,}$, $\cdots$, d_{IX}, and note their intersections with the corresponding horizontal lines, we shall obtain the points d''', d'''', $\cdots$, d^{IX}, other points in the vertical projection of the helix.

215. To assume a Point upon the Helix. See Fig. 86. Assume the horizontal projection of the point anywhere upon the horizontal projection of the helix, and through this point draw a straight line perpendicular to G–L to intersect the vertical projection of the helix in the required vertical projection of the point.

216. To draw a Rectilinear Tangent to a Helix at a Point assumed on the Curve.

Analysis. Since the generating point of a helix retains a constant distance from the axis, the curve may be traced upon the surface of a right circular cylinder whose axis is the axis of the helix.

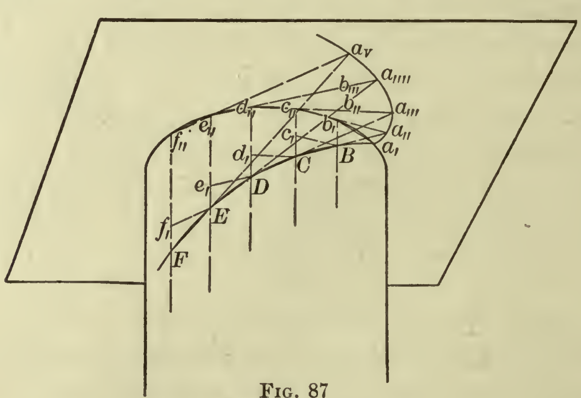

Fig. 87

Fig. 87 represents pictorially a right circular cylinder with a portion of a helix, $a_{,}$–B–C–D–E–F, traced upon the surface.

Assuming the axis of the cylinder perpendicular to H, the portion of the base $a_{,}$–$b_{,}$–$c_{,,}$–$d_{,,}$–$e_{,,}$–$f_{,,}$ represents the horizontal projection of the portion of the helix in question.

Let $a_{,}$ represent the point in which the helix pierces H. Let B represent the point consecutive to $a_{,}$ (greatly magnified in its distance from $a_{,}$), and let $b_{,}$ represent the horizontal projection of B. Let C represent the point consecutive to B and let $c_{,,}$ represent the horizontal projection of C.

In the same way let D, E, F, etc., represent consecutive points of the curve and let $d_{,,}$, $e_{,,}$, $f_{,,}$, etc., represent respectively their horizontal projections.

Connect $a_{,}$ and B by a straight line; also connect $a_{,}$ and $b_{,}$ by a straight line. These two lines together with the horizontal projecting line of B form a right-angled triangle in which $a_{,}–b_{,}$ is the horizontal projection of $a_{,}–B$.

Practically speaking, the two lines $a_{,}–B$ and $a_{,}–b_{,}$ are identical with the arcs of which they are chords. For this reason the angle $B–a_{,}–b_{,}$ measures the slope of the helix, that is, the constant inclination of the curve to H, or the angle which a rectilinear tangent to the helix at any point makes with H.

Connect B and C by a straight line; also connect $b_{,}$ and $c_{,,}$ by a straight line. These two lines together with the two projecting lines $B–b_{,}$ and $C–c_{,,}$ form a quadrilateral in which $B–C$ will make the same angle with H as that made by $a_{,}–B$, since the uniform motions of the generating point give to each elementary portion of the helix the same inclination to H.

If the plane of the triangle $a_{,}–B–b_{,}$ be revolved about $B–b_{,}$ as an axis until it coincides with the plane of the quadrilateral $B–c_{,,}$, the line $a_{,}–b_{,}$ will take the position $a_{,,}–b_{,}$, which is a continuation of $c_{,,}–b_{,}$; and the line $a_{,}–B$ will take the position $a_{,,}–B$, which is a continuation of $C–B$.

If the plane of the triangle $a_{,,}–C–c_{,,}$ be revolved about $C–c_{,,}$ as an axis until it coincides with the plane of the quadrilateral $C–d_{,,}$, the line $a_{,,}–b_{,}–c_{,,}$ will take the position $a_{,,,}–b_{,,}–c_{,,}$, which is a continuation of $d_{,,}–c_{,,}$; and the line $a_{,,}–B–C$ will take the position $a_{,,,}–C$, which is a continuation of $D–C$.

The line $a_{,,}–B$ is tangent to the helix at the point B, since it contains B and its consecutive point C. The line $a_{,,}–b_{,}$ is the horizontal projection of this tangent and is itself tangent to the horizontal projection of the helix at the point $b_{,}$. The tangent $a_{,,}–B$ pierces H at $a_{,,}$, at a distance from $b_{,}$ equal to the rectification of the arc $a_{,}–b_{,}$.

Again, $a_{,,,}–C$ is tangent to the helix at the point C, since it contains C and its consecutive point D. The line $a_{,,,}–b_{,,}–c_{,,}$ is the horizontal projection of this tangent and is itself tangent to the

horizontal projection of the helix at the point $c_{\prime\prime}$. The tangent $a_{\prime\prime\prime}$–C pierces H at $a_{\prime\prime\prime}$, at a distance from $c_{\prime\prime}$ equal to the rectification of the arc $a_{\prime}$–$b_{\prime}$–$c_{\prime\prime}$.

That portion of the tangent $a_{\prime\prime}$–B between the points $a_{\prime\prime}$ and B is the rectified length of the curve $a_{\prime}$–B, and that portion of the tangent $a_{\prime\prime\prime}$–C between the points $a_{\prime\prime\prime}$ and C is the rectified length of the curve $a_{\prime}$–B–C.

We have now considered three consecutive points, $a_{\prime}$, B, and C, of the helix, and it will be evident that the same process of reasoning may be applied indefinitely to the consecutive points of the curve.

We may conclude, then, that the horizontal projection of a rectilinear tangent to a helix at any point on the curve, provided the axis of the helix is perpendicular to H, will be tangent to the horizontal projection of the helix at the horizontal projection of the point of tangency; also that the tangent itself will pierce H upon the horizontal projection of the tangent and at a distance from the horizontal projection of the point of tangency equal to the rectification of that portion of the horizontal projection of the helix between the point in which the helix pierces H and the horizontal projection of the point of tangency.

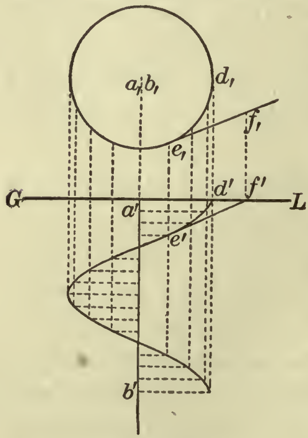

Fig. 88

By use of these principles we can draw the two projections of a rectilinear tangent to the helix when the axis of the helix is assumed perpendicular to H.

Construction. Let the helix be represented as in Fig. 88. Assume any point, as E, upon the curve (see Section 215). At $e_{\prime}$ draw $e_{\prime}$–$f_{\prime}$ tangent to the horizontal projection of the helix. Upon this tangent lay off from $e_{\prime}$ a distance $e_{\prime}$–$f_{\prime}$ equal to the rectification of the arc $e_{\prime}$–$d_{\prime}$, where $d_{\prime}$ is the point in which the helix pierces H, and where $e_{\prime}$ is the horizontal projection of the point of tangency. The line $e_{\prime}$–$f_{\prime}$ is the horizontal projection of the required tangent, and $f_{\prime}$ is the point in which this tangent pierces H. The vertical

projection of F is at f', on G–L, and f'–e' is the vertical projection of the required tangent.

By this process the position of the vertical projection of the tangent to the helix is determined by two points, e' and f', and the unsatisfactory task of drawing a rectilinear tangent to an irregular curve is avoided.

217. Problem 144. *Represent a left-handed helix whose radius is* 4 *and whose pitch is* 8.

218. Problem 145. *Find the point in which the right-handed helix whose radius is* 4 *and pitch* 10 *intersects a horizontal plane* 4 *above H.*

219. Problem 146. *Find the point in which the right-handed helix whose radius is* 4 *and pitch* 9 *intersects a plane containing the axis and making an angle of* 45 *degrees with V.*

CHAPTER VIII

GENERATION AND CLASSIFICATION OF SURFACES

220. Generation of Surfaces. We may regard a surface as the generation resulting from the movement of a line which in the course of the generation occupies an infinite number of consecutive positions at infinitely small distances apart. The moving line is called the *generatrix*, and the various positions which it occupies during the generation are called *elements* of the surface.

The portion of the surface generated by the line while moving from one position to its consecutive position is called an *elementary surface*, and while in theory the distance between the two consecutive elements must be taken into account, practically speaking the two elements may be regarded as one and the same.

221. Classification of Surfaces. The character of the surface generated will depend both upon the character of the generatrix and upon the nature of its motion.

Depending upon the character of the generatrix we have two classes of surfaces: first, those which are generated by straight lines, or those which have rectilinear elements; and second, those which are generated by curved lines, known as *surfaces of double curvature.*

Depending upon the nature of the motion of a rectilinear generatrix we have (1) the *plane* which may be generated by one straight line moving in such a way as to touch another straight line and always remaining parallel to its first position; (2) the *single curved surface* which may be generated by a straight line moving in such a way that any two of its consecutive positions shall be in the same plane; and (3) the *warped surface* which may be generated by a straight line moving in such a way that no two of its consecutive positions shall be in the same plane.

222. Single Curved Surfaces. When the rectilinear generatrix moves in such a way that all its positions are parallel but no three

of its consecutive positions lie in the same plane, the surface is a *cylindrical surface.*

When the rectilinear generatrix moves in such a way that all its positions pass through a common point and no three of its consecutive positions lie in the same plane, the surface is a *conical surface.*

When the rectilinear generatrix moves in such a way that consecutive positions intersect two and two and at the same time in such a way that no three consecutive positions lie in the same plane, the surface is a *convolute.*

223. Surfaces of Revolution. Surfaces of revolution are those which may be generated by the revolution of a line about a straight line as an axis.

If the generatrix is a straight line and is parallel to the axis, the surface is a *cylindrical surface of revolution*, — a single curved surface.

If the generatrix is a straight line and intersects the axis obliquely, the surface is a *conical surface of revolution*, — a single curved surface.

If the generatrix is a straight line and does not lie in the same plane with the axis, the surface is a *warped surface of revolution*, since from the nature of the generation consecutive elements of the surface cannot lie in the same plane.

If the generatrix is a curved line, as it will be in all cases save those mentioned above, the surface will be one of double curvature, or a *double curved surface of revolution.*

224. Double Curved Surfaces of Revolution. If the generatrix of a double curved surface of revolution is the circumference of a circle and the axis is a diameter of the circle, the surface generated is that of a *sphere.*

If the generatrix is the curve of an ellipse and the axis is one of the axes of the ellipse, the surface generated is that of the *ellipsoid of revolution.* The ellipsoid of revolution is called a *prolate* or an *oblate spheroid* according as the long or the short axis is used.

If the generatrix is the curve of a parabola and the axis is the axis of the parabola, the surface generated is that of the *paraboloid of revolution.*

If the generatrix is the curve of a hyperbola and the axis is one of the axes of the hyperbola, the surface generated is that of the *hyperboloid of revolution.*

225. Representation of Surfaces. Surfaces which exist within definite limitations are usually represented by projecting upon *H* and *V* the limiting lines as seen from the two principal standpoints of projection. Other surfaces are too irregular in their formation to be represented in this way, and all that is attempted is to represent by projection a sufficient number of the elements of the surface to reveal the character of some small portion of the surface under consideration.

226. Tangents to Surfaces. A straight line is tangent to a surface at a given point when it is tangent to a line of the surface at that point.

A plane is tangent to a surface at a given point when it contains all the rectilinear tangents to the surface at that point. In other words, if a plane is tangent to a surface, and any cutting plane be passed through the point of tangency, the cutting plane will cut from the surface a line, and from the tangent plane a straight line, tangent to the first line at the point of tangency.

Therefore, to draw a plane tangent to a surface at a given point, draw two rectilinear tangents to the surface at this point and determine their plane.

If the surface has rectilinear elements, the tangent plane must contain the rectilinear element passing through the point of tangency, since the rectilinear tangent to a rectilinear element is the element itself. This element is the element of tangency.

If the surface is of single curvature, we may say that a plane is tangent to the surface at a given point when it represents the plane in which the generatrix is moving at the instant in which it passes through the point of tangency.

For this reason we may say that a plane is tangent to a single curved surface at a given point when it contains both the element through the point of tangency and its consecutive one. If through the element containing the point of tangency we pass a secant plane, it will cut the surface in two distinct lines, one of which is the element through the point of tangency and the other another

line somewhat removed from the first. If this secant plane be revolved about the element of tangency as an axis toward the position of the tangent plane, the element of tangency will remain stationary and the second line will gradually approach the position of the first; and when the second line takes the position consecutive to the element of tangency, or practically the position of the element of tangency itself, the secant plane will take the position of the tangent plane.

If a plane is tangent to a single curved surface at a given point, it will be tangent to the surface all along the rectilinear element containing the point of tangency. For if through any point of this element a cutting plane be passed oblique to the elements, it will cut from the consecutive element (which is also a line of the tangent plane) a point consecutive to the assumed point.

These two consecutive points lie in the tangent plane and at the same time lie on the line cut from the surface by the oblique plane. A straight line through these two points is tangent to the line cut from the surface at the assumed point, and lies in the tangent plane. The tangent plane is then tangent to the surface at this point, since it contains two rectilinear tangents to the surface at this point.

If then a plane is tangent to a single curved surface, any plane passed oblique to the element of tangency will cut from the tangent plane a straight line which will be tangent to the line cut from the surface at the point where the element of tangency intersects the oblique plane.

Two surfaces are tangent to each other when they are both tangent to the same surface at a common point, or when planes passed through their point of contact cut from the two surfaces lines which are tangent to each other at the point of contact.

227. Normals to Surfaces. A straight line is normal to a surface at a given point when it is perpendicular to the plane which is tangent to the surface at that point.

A plane is normal to a surface at a given point when it contains the rectilinear normal to the surface at that point.

Evidently there can be but one rectilinear normal to a surface at a given point, but an infinite number of plane normals.

228. The Development of Surfaces. By the development of a surface we mean its rectification, or its measure or appearance when laid out on a plane.

Just as a curved line is rectified by rolling the curve out on a rectilinear tangent to the curve, so a surface is developed by rolling the surface out on a plane tangent to the surface.

In cases of prisms and pyramids, or of any surfaces made up of plane faces, the plane of one of the faces is taken as the plane of development, and the successive faces are brought into coincidence with the plane of development by revolving them one after another about the edges as axes.

In cases of cylinders and cones, or other surfaces of single curvature, a plane tangent to the surface along some element is taken as the plane of development, and the surface is then rolled out element after element, peeling off, as it were, the outer coating of the surface and spreading it out on the plane.

The development of a surface may be used as a templet, or pattern, for cutting out a plane surface form, which by a process converse to that employed in development may be made to take the shape of the original surface.

CHAPTER IX

REPRESENTATION OF SURFACES WITH PLANE FACES

229. Surfaces consisting of Plane Faces. Such surfaces are usually placed, for ease of representation, in such a position that

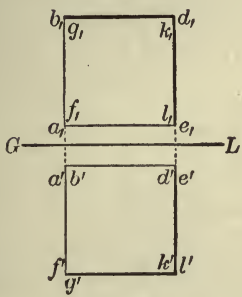

FIG. 89

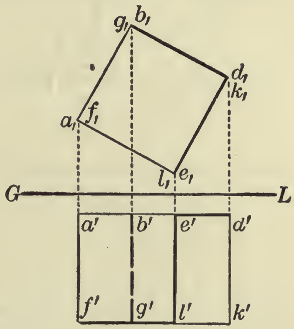

FIG. 90

their principal faces are either perpendicular to or parallel to the planes of projection.

Fig. 89 represents a cube in the third quadrant with two of its faces parallel to *H* and with two of its faces parallel to *V*.

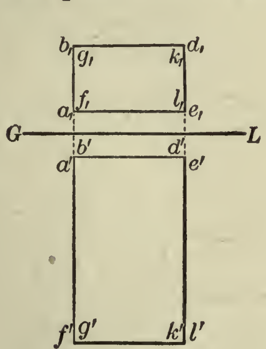

FIG. 91

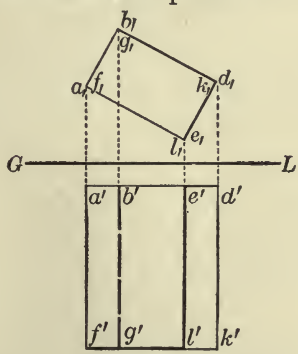

FIG. 92

Fig. 90 represents a cube, two of whose faces are parallel to *H* but whose vertical faces are oblique to *V*.

Fig. 91 represents a rectangular prism with bases parallel to H and with two faces parallel to V.

Fig. 92 represents a rectangular prism whose edges are perpendicular to H and whose faces are oblique to V.

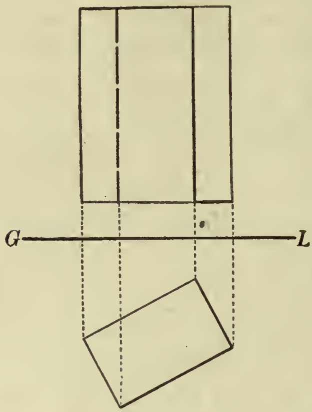

FIG. 93

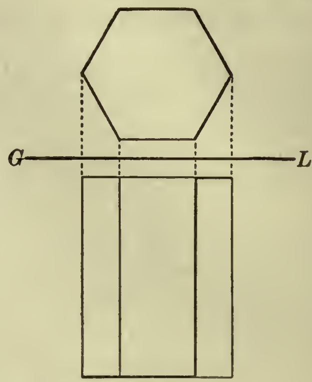

FIG. 94

Fig. 93 represents a rectangular prism whose edges are perpendicular to V and whose faces are oblique to H.

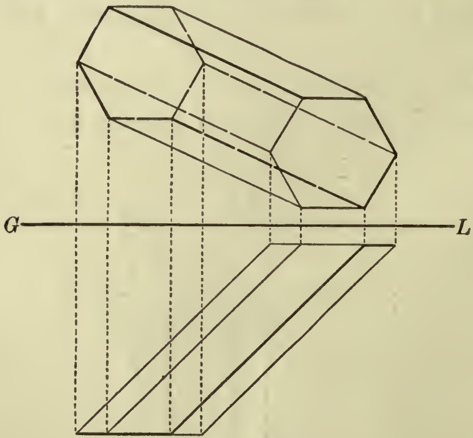

FIG. 95

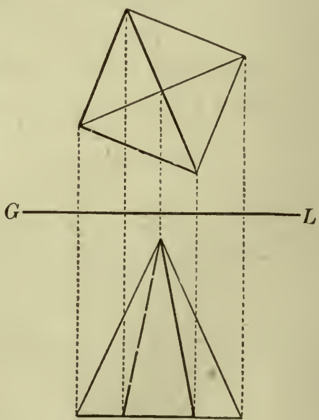

FIG. 96

Fig. 94 represents a hexagonal prism with bases parallel to H and with edges perpendicular to H.

Fig. 95 represents an oblique hexagonal prism with bases parallel to H.

Fig. 96 represents a square pyramid with base parallel to H and with axis perpendicular to H.

Fig. 97 represents a pentagonal pyramid with base parallel to H and with axis perpendicular to H.

Fig. 98 represents the frustum of a square pyramid with bases parallel to H.

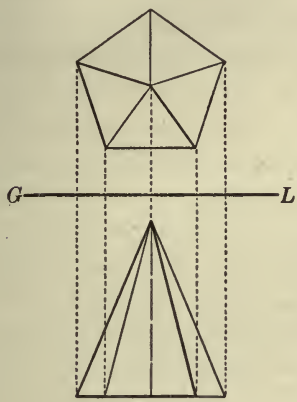

Fig. 97

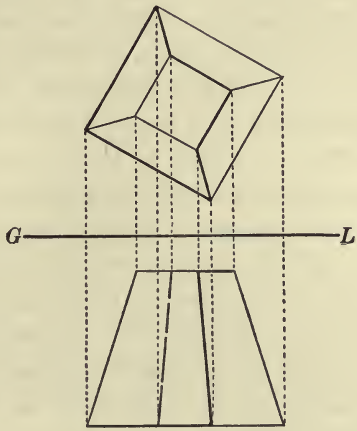

Fig. 98

230. Shade Lines. Objects in nature, as a rule, are exposed to some source of light so that some portions of their surface are in the light and other portions are in the dark.

The sun is usually taken as the source of light, and on account of its great distance from the earth we may safely consider such solar rays as fall upon terrestrial objects of finite dimensions as parallel.

The source of light is conventionally assumed in such a position that the rays shall be parallel to that diagonal of a cube (the cube resting on H in the first quadrant with one face coincident with V) which slopes downward to the right toward V.

Under these conditions the horizontal and vertical projections of a ray of light will each make an angle of 45 degrees with G–L.

When the position of the object is definitely known it will be easy to determine which portions of the surface are in the light and which portions are in the dark.

Those lines on a surface which separate light portions from dark portions are called *shade lines*, and in the drawing, for purposes of clearness, are represented a little heavier than the ordinary line.

In Fig. 89, remembering the direction taken by rays of light, it is evident that the upper, the left-hand, and the front faces of the cube are in the light and that all the others are in the dark.

Therefore the shade lines in this case are B–D, D–E, E–L, L–F, F–G, and G–B, and are therefore represented by heavy lines.

In Fig. 90 the shade line B–G is invisible and is therefore represented by a heavy broken line.

When two lines of the surface, one a shade line and the other an ordinary line, are projected upon the same line, it is customary to give preference to that line which is visible. For example, in Fig. 89 the lines A–E and F–L have a common horizontal projection $a_{,}$–$e_{,}$, but the line A–E is the visible line, and inasmuch as it is not a shade line the projection $a_{,}$–$e_{,}$ is made light.

In Fig. 90, B–D and G–K have a common horizontal projection, but since B–D is the visible line and is a shade line, $b_{,}$–$d_{,}$ is made a heavy line.

Following the directions now given, it will be easy to determine the shade lines in the remaining diagrams.

CHAPTER X

REPRESENTATION OF SINGLE CURVED SURFACES

231. Cylindrical Surfaces and the Cylinder. The cylindrical surface is a single curved surface which may be generated by a moving straight line which during the movement touches a given curved line and always remains parallel to its first position (see Section 222).

The generating line is called the *generatrix*, the curved line is called the *directrix*, and the various positions occupied by the generatrix are called the *rectilinear elements* of the surface.

The intersection of a cylindrical surface by any plane which is not parallel to the elements is called a *section*, or *base*, of the surface. When this plane is taken perpendicular to the elements, the section is called a *right section*, and according to the nature of this section cylindrical surfaces are classified as *circular*, *elliptical*, *parabolic*, *hyperbolic*, *etc.*

The section of a cylindrical surface made by the plane H will often serve as a convenient base.

A plane parallel to the elements of a cylindrical surface and cutting the surface will cut the surface in elements, since all the elements of such surfaces are parallel.

If the base of a cylindrical surface is a closed curve, like a circle or an ellipse, the space inclosed by the cylindrical surface is called a *cylinder*.

The straight line through the center of the base and parallel to the elements of the surface is called the *axis* of the cylinder.

If one of two parallel straight lines is revolved about the other as an axis, the surface so generated is cylindrical, and the cylinder so generated is a *cylinder of revolution* whose right section is a circle.

232. To represent the Cylinder. The cylinder is usually represented by projecting one of its bases, which may be assumed

anywhere but which must not be regarded as limiting the surface, and by projecting its extreme elements, that is, those elements which from the observer's position appear to limit the surface.

CASE 1. *To represent a circular cylinder whose axis is perpendicular to H and whose bases are right sections.* See Fig. 99.

CASE 2. *To represent a circular cylinder whose axis is perpendicular to V and whose bases are right sections.* See Fig. 100.

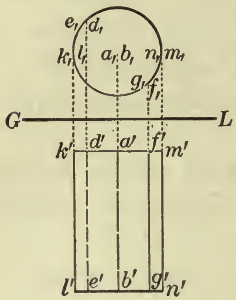

FIG. 99

CASE 3. *To represent the cylinder when its base is on H.* See Fig. 101. Let the circle whose center is $a_{,}$ represent the base on H, and let A–B represent the axis. Tangent to the circular base and parallel to $a_{,}$–$b_{,}$ draw the straight lines $d_{,}$–$e_{,}$ and $f_{,}$–$g_{,}$. These lines represent the horizontal projections of the extreme or limiting elements, as seen from the observer's standpoint while projecting on H, since no elements can occupy positions at greater distance from the horizontal projecting plane of A–B. The vertical projections of D and F are d' and f' respectively. Through d' and f' and parallel to a'–b' draw the straight lines d'–e' and f'–g', to represent the vertical projections of these elements.

Tangent to the circular base and perpendicular to G–L draw the straight lines $k_{,}$–k' and $m_{,}$–m'. Through k' and m' and parallel to a'–b' draw the straight lines k'–l' and m'–n'. These lines represent the vertical projections of the extreme or limiting elements, as seen from the observer's standpoint while projecting on V, since no elements can occupy positions at greater distance from the vertical projecting plane of A–B. Through $k_{,}$ and $m_{,}$ and parallel to $a_{,}$–$b_{,}$ draw the straight lines $k_{,}$–$l_{,}$ and $m_{,}$–$n_{,}$, to represent the horizontal projections of these elements.

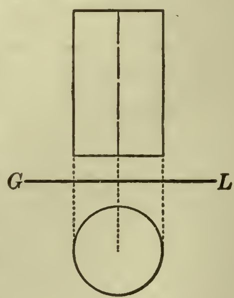

FIG. 100

The outline $e_{,}$–$d_{,}$–$k_{,}$–$f_{,}$–$g_{,}$ represents the horizontal projection of the cylinder, and the outline l'–k'–m'–n' represents its vertical projection.

It will be evident that if the dimensions or position of the cylinder be changed, the projections of the cylinder will also be changed. It will also be evident that a cylinder of definite dimension and position will have definite projections fully determining the cylinder.

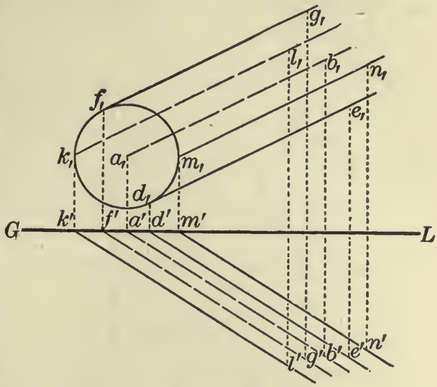

FIG. 101

It must be remembered that in assuming the circular base and the inclination of the axis, both at random, we have represented a cylinder at random and not one of definite dimension and position.

CASE 4. *To represent the cylinder when the plane of its base is oblique to H but perpendicular to V.* In Fig. 102 the circle whose center is a_1 represents the horizontal projection of the base; the straight line k'–m' included between the two vertical lines k_1–k' and m_1–m', each drawn tangent to the circle whose center is a_1, represents the vertical projection of the base, and A–B represents the axis.

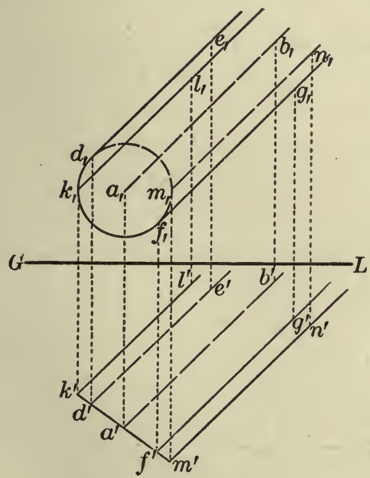

FIG. 102

The horizontal projections of the extreme elements D–E and F–G can now be determined as in the previous case. The lines k'–l' and m'–n' represent the vertical projections of the highest and lowest, and therefore the extreme, elements as seen from the observer's standpoint while projecting on V.

233. To assume a Rectilinear Element of the Cylinder. Let the cylinder be represented with a circular base on H, as shown in Fig. 103. Assume any point, as E, in the circumference of the

base. Through $e_{\prime}$ draw $e_{\prime}$–$o_{\prime}$ parallel to $a_{\prime}$–$b_{\prime}$, and through e' draw e'–o' parallel to a'–b'. E–O is the required element.

234. To assume a Point upon the Surface of the Cylinder. Let the cylinder be represented as in Fig. 103. Assume at random the horizontal projection $o_{\prime}$ of the point. Draw $o_{\prime}$–$d_{\prime}$–$e_{\prime}$ parallel to $a_{\prime}$–$b_{\prime}$, to represent the horizontal projection of the element containing the point O. This element pierces H either at $d_{\prime}$ or at $e_{\prime}$ according as O is assumed on the upper or on the lower surface of the cylinder. Assuming the point on the lower surface of the cylinder, the element in question will be vertically projected at e'–o', and the point O must be vertically projected at o' on the straight line through $o_{\prime}$ perpendicular to G–L.

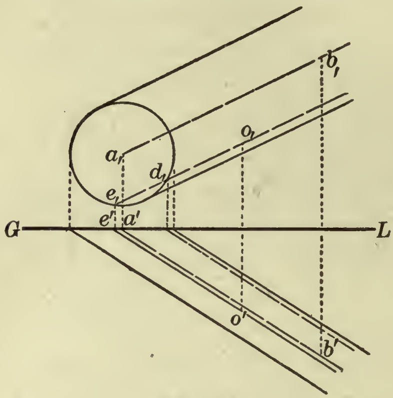

FIG. 103

235. Problem 147. *Given a cylinder whose axis is oblique to H and V and whose base is a circle in V; required to draw the two projections of the cylinder and to assume a point upon the surface.*

236. Problem 148. *Given a cylinder whose right section is a circle and whose axis is in such a position that its horizontal projection is inclined* 30 *degrees to G–L and its vertical projection is inclined* 60 *degrees to G–L; required to draw the two projections of the cylinder and to assume a point upon the surface.*

237. Problem 149. *Given a cylinder whose right section is a circle and whose axis is parallel to G–L; required to draw the two projections of the cylinder and to assume a point upon the surface.*

238. Problem 150. *Given a cylinder whose base is a circle on H and whose axis is in a profile plane and oblique to H; required to draw the two projections of the cylinder and to assume a point upon the surface.*

239. Conical Surfaces and the Cone. The conical surface is a single curved surface which may be generated by a moving straight line which during the movement touches a given curved line and constantly passes through a given fixed point (see Section 222).

The generating line is called the *generatrix*, the curved line is called the *directrix*, the fixed point is called the *vertex*, and the various positions occupied by the generatrix are called *rectilinear elements* of the surface.

It is evident from the nature of the generation of the conical surface that there will be generated simultaneously, on opposite sides of the vertex, two equivalent portions of the surface. These portions of the surface are called *nappes* of the surface.

It is also evident that if the vertex be removed to an infinite distance from the directrix, the conical surface will become cylindrical.

The intersection of a conical surface by any plane not containing the vertex is called a *section*, or *base*, of the surface. When this plane is taken perpendicular to the axis* the section is called a *right section*, and conical surfaces are classified according to the nature of this section as *circular*, *elliptical*, *parabolic*, *hyperbolic*, etc.

The section of a conical surface made by the plane H will often serve as a convenient base.

A plane containing the vertex of a conical surface and intersecting the surface will cut the surface in elements, since all the elements of such surfaces pass through the vertex.

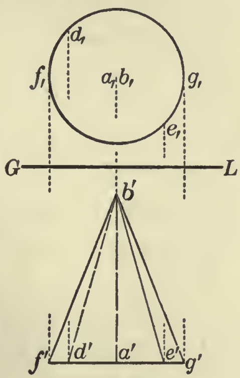

FIG. 104

If the base of a conical surface is a closed curve, like a circle or an ellipse, the space inclosed by the conical surface is called a *cone*, and the straight line connecting the center of the base with the vertex is called the *axis* of the cone.

240. To represent the Cone. A cone is usually represented by projecting the vertex, a base, and the extreme or limiting elements.

CASE 1. *To represent a circular cone whose axis is perpendicular to H and whose base is a right section.* See Fig. 104, in which $A–B$ represents the axis of the cone and in which $F–D–G–E$ represents the circular base.

* For definition of axis, see few lines below.

CASE 2. *To represent a circular cone whose axis is perpendicular to V and whose base is a right section.* See Fig. 105.

CASE 3. *To represent the cone when its base is on H and its axis is oblique to H and V.* See Fig. 106.

Let the circle whose center is $a_{\prime}$ represent the base on H, let B represent the vertex, and let A–B represent the axis.

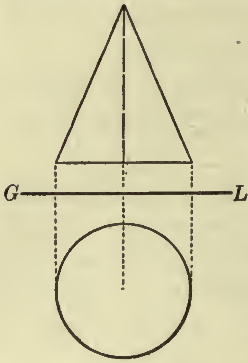

FIG. 105

Through the point $b_{\prime}$ and tangent to the circle whose center is $a_{\prime}$ draw $b_{\prime}$–$d_{\prime}$ and $b_{\prime}$–$e_{\prime}$. These lines represent the horizontal projections of the extreme or limiting elements as seen from the observer's standpoint while projecting on H, since no elements can occupy positions at greater distance from the horizontal projecting plane of A–B. The vertical projections of D and E are d' and e' respectively, and the lines b'–d' and b'–e' are the vertical projections of these elements.

Tangent to the circular base and perpendicular to G–L draw $f_{\prime}$–f' and $g_{\prime}$–g'. Connect f' and b', also connect g' and b', by straight lines. These lines represent the vertical projections of the extreme or limiting elements as seen from the observer's standpoint while projecting on V, since no elements can occupy positions at greater distance from the vertical projecting plane of A–B. The lines $f_{\prime}$–$b_{\prime}$ and $g_{\prime}$–$b_{\prime}$ represent the horizontal projections of these elements.

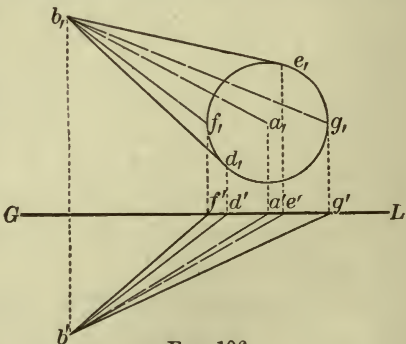

FIG. 106

The outline $b_{\prime}$–$d_{\prime}$–$g_{\prime}$–$e_{\prime}$–$b_{\prime}$ represents the horizontal projection of the cone, and the outline b'–f'–g'–b' represents the vertical projection of the cone.

It will be evident that if the dimensions or position of the cone be changed, the projections of the cone will also be changed. It will also be evident that a cone of definite dimension and position will have definite projections fully determining the cone.

It must be remembered that in assuming the circular base and the vertex, both at random, we have represented a cone at random and not one of definite dimension and position.

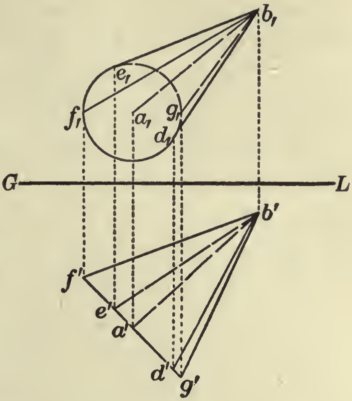

Fig. 107

Case 4. *To represent the cone when the plane of its base is oblique to H but perpendicular to V.* See Fig. 107. The circle whose center is $a_{\prime}$ represents the horizontal projection of the base, and the straight line f'–g' included between the two vertical lines $f_{\prime}$–f' and $g_{\prime}$–g', each drawn tangent to the circle whose center is $a_{\prime}$, represents the vertical projection of the base. B represents the vertex and A–B represents the axis.

The horizontal projections of the extreme elements B–D and B–E may now be determined as in the previous case.

The lines b'–f' and b'–g' represent the vertical projections of the highest and the lowest elements, and therefore the extreme elements, as seen from the observer's standpoint while projecting on V.

241. To assume a Rectilinear Element of the Cone. Let the cone be represented with a circular base on H, as shown in Fig. 108.

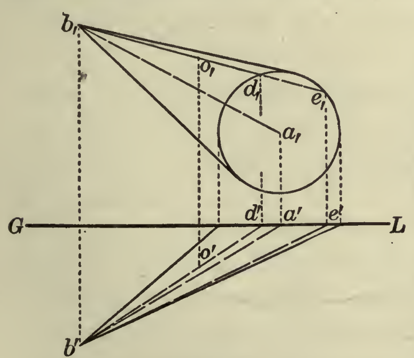

Fig. 108

Assume any point, as D, on the circumference of the base. Connect D and B by a straight line. B–D, horizontally projected in $b_{\prime}$–$d_{\prime}$ and vertically projected in b'–d', is the required element.

242. To assume a Point upon the Surface of the Cone. Let the cone be represented as in Fig. 108. Assume the horizontal projection $o_{\prime}$ of the point at random. Draw $b_{\prime}$–$o_{\prime}$–$d_{\prime}$–$e_{\prime}$ to represent the horizontal projection of the element containing the point O. This element pierces H either

at $d_{,}$ or at $e_{,}$ according as O is assumed on the upper or on the under surface of the cone.

Assuming the point on the upper surface of the cone, the element in question will be vertically projected at b'–d', and the point O must be vertically projected at o' on the straight line through $o_{,}$ perpendicular to G–L.

243. Shade Lines. Shade lines upon cylindrical and conical surfaces, since such lines are purely imaginary elements of the surface, unless they occur at the intersection of base and surface, are best not represented by heavy lines.

When the plane of the base of a cylinder or cone is perpendicular to the plane of projection, the projection of the base upon this plane of projection is a straight line, and as a rule will represent a line in space, which is partially a shade line and partially not.

For the sake of appearance the projection of the base under these conditions will be drawn either wholly a heavy line or wholly a light line according as the portion which should be represented as a shade line exceeds or does not exceed in length that portion which should be represented as a light line.

In Fig. 99, remembering the direction taken by the rays of light, it is evident that the upper base of the cylinder and that portion of the surface to the left and limited by the two elements D–E and F–G will be in the light, and that the remainder of the surface will be in the dark. Therefore the only shade lines to be represented in this case are D–M–F and E–L–G.

When projecting on H, D–M–F is visible and its horizontal projection $d_{,}$–$m_{,}$–$f_{,}$ is drawn as a heavy line. When projecting on V the portion M–F of the shade line D–M–F and the portion L–G of the shade line E–L–G are visible. Since, then, only the small portion m'–f' of the vertical projection of the upper base should be represented as a heavy line, the whole line is made a light line. Since the larger portion l'–g' of the vertical projection of the lower base should be represented as a heavy line, the whole line is made a heavy line.

In Fig. 104 the shade line to be represented is approximately D–F–E. Since the larger portion f'–e' of the vertical projection of the base should be made a heavy line, the whole line f'–g' is made a heavy line.

244. Problem 151. *Given a cone whose axis is oblique to H and V and whose base is a circle on V; required to draw the two projections of the cone and to assume a point upon the surface.*

245. Problem 152. *Given a cone whose axis is perpendicular to H, whose vertex is* 1 *unit below H, and whose right section at the distance of* 8 *units below H is a circle* 6 *units in diameter; required to draw the two projections of the cone and to assume a point upon the surface.*

246. Problem 153. *Given a cone whose base is a circle on H and whose axis is in a profile plane and oblique to H; required to draw the two projections of the cone and to assume a point upon the surface.*

247. The Convolute. The convolute is a single curved surface which may be generated by a straight line moving tangentially to a curve of double curvature. The generating line is called the *generatrix*, the curved line is called the *directrix*, and the various positions occupied by the generatrix are called the *rectilinear elements* of the surface.

Since a rectilinear tangent to a curved line contains two consecutive points of the curve, two consecutive tangents must have a point in common and therefore intersect. The consecutive elements of the convolute, then, will intersect, and since the directrix is of double curvature, only those elements which are consecutive will, in general, intersect.

If the directrix is a helix, the convolute is called a *helical convolute.*

248. To represent the Helical Convolute. The helical convolute is represented in Fig. 87, where for purposes of clearness the distances between consecutive positions of the generatrix are greatly magnified.

The curve $a_{\prime}–B–C–D–E–F$ represents the helical directrix. The straight lines $a_{\prime\prime}–B–C$, $a_{\prime\prime\prime}–C–D$, $a_{\prime\prime\prime\prime}–D–E$, etc., represent positions of the generatrix or rectilinear elements of the surface.

It will be noticed here, as stated above, that only consecutive elements intersect; for example, $a_{\prime\prime\prime}–C–D$ intersects its preceding consecutive element, $a_{\prime\prime}–B–C$, at C; it also intersects its succeeding consecutive element, $a_{\prime\prime\prime\prime}–D–E$, at D; but it does not intersect the element $a_{V}–E–F$, which is not consecutive to it.

If we conceive a large number of these tangents or elements of the surface to be drawn at small distances apart, we may form an idea of the character of the surface.

The curved line $a_{,}$–$a_{,,}$–$a_{,,,}$–$a_{,,,,}$–, etc., traced through the various points in which the elements pierce H, is the intersection of the surface with H and may be taken as the base of the surface.

From the method by which these points $a_{,,}$, $a_{,,,}$, $a_{,,,,}$, etc., are found (see Section 216), it will be seen that the base of the helical convolute, in case the axis of the helical directrix is taken perpendicular to H, is the involute of that circle which represents the horizontal projection of the directrix.

In Fig. 87 only that portion of the surface extending between the helical directrix and the base on H is considered.

Since the generatrix extends without limit in both directions from the point of tangency on the helix, there will be generated simultaneously on opposite sides of the helical directrix two distinct portions of the surface. These two portions of the surface are called *nappes* of the surface, and their line of separation, which is the helical directrix, is called the *edge of regression.*

From the nature of the helical directrix and the relation of the generatrix to the directrix it is evident that if the generation be extended beyond one circuit of the axis, the surface will consist of a series of overlapping surfaces whose base or intersection with H will have the form of a spiral.

249. To assume a Rectilinear Element of the Helical Convolute. Draw a rectilinear tangent to the helical directrix (see Section 216).

250. To assume a Point upon the Surface of the Helical Convolute. First assume an element of the surface and then assume a point upon the element.

CHAPTER XI

REPRESENTATION OF WARPED SURFACES

251. The warped surface may be generated by a straight line moving in such a way that its consecutive positions do not remain in the same plane. Evidently there can be as many warped surfaces as there are distinct laws restricting the motion of straight lines in this way.

252. The Warped Surface with Two Linear Directrices and a Plane Directer. The warped surface with two linear directrices and a plane directer is generated by the movement of a straight line which constantly touches two linear directrices and remains parallel to a plane directer.

253. The Hyperbolic Paraboloid. The hyperbolic paraboloid is a warped surface with two rectilinear directrices and a plane directer.

This surface is represented by locating the two rectilinear directrices, the plane directer, and a sufficient number of the rectilinear elements, or positions occupied by the generatrix, to reveal the character of that portion of the surface under consideration.

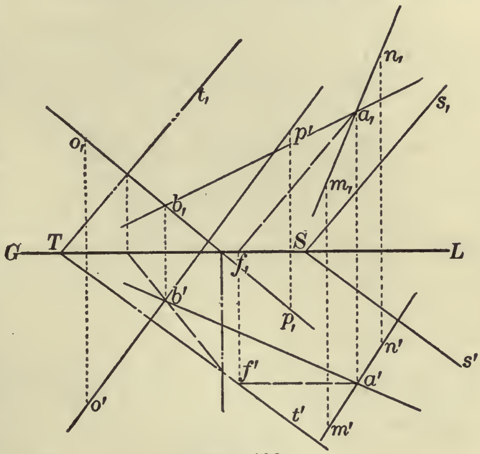

FIG. 109

254. To assume a Rectilinear Element of the Hyperbolic Paraboloid. See Fig. 109. Let $M–N$ and $O–P$ represent the rectilinear directrices, and let S represent the plane directer.

Assume any point, as A, upon the directrix $M–N$, and through A, by Section 172, draw the plane T parallel to S. By Section 151 find the point B in which the directrix $O–P$ intersects S, and

connect A and B by a straight line. A–B is an element of the surface, since it touches the two directrices and lies in the plane T which is parallel to the plane directer.

255. To assume a Point upon the Surface of the Hyperbolic Paraboloid.

Analysis 1. First assume an element of the surface and then assume a point upon this element.

Analysis 2. Assume at random the horizontal projection of the point. Through the horizontal projecting line of the point draw a plane and find the intersection of this plane with the surface. This intersection must contain the point upon the surface, and the vertical projection of the point must lie both upon the vertical projection of this intersection and upon the vertical projection of the horizontal projecting line of the point.

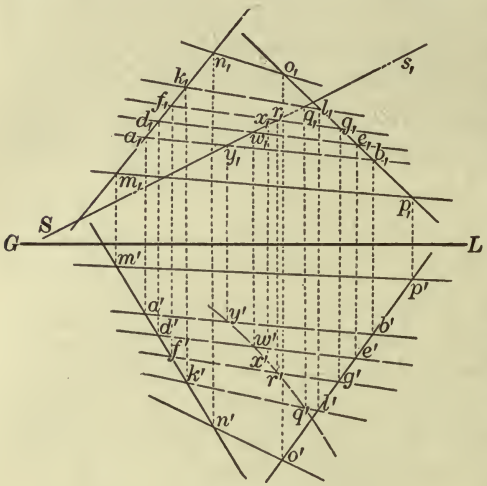

Fig. 110

Construction. See Fig. 110. Let M–N and O–P represent the two rectilinear directrices, and let N–O and M–P represent two rectilinear elements of the surface.

Any plane parallel to the two lines N–O and M–P is the plane directer of the surface.

Assume at random x_1 as the horizontal projection of the required point. Through the horizontal projecting line of X pass any plane, as S.

By Section 254 assume a number of rectilinear elements of the surface in the vicinity of X. These elements are A–B, D–E, F–G, K–L, etc. These elements, beginning with A–B, intersect the plane S in the points Y, W, R, Q, etc. The line Y–W–R–Q is the intersection of S and the warped surface, and must contain X.

The point x', the point in which a straight line through $x_{,}$ perpendicular to G–L intersects y'–w'–r'–q', is the required vertical projection.

The greater the number of rectilinear elements assumed, the more accurately will the point x' be located.

256. The Warped Surface with two Curvilinear Directrices and a Plane Directer. The warped surface with two curvilinear directrices and a plane directer is generated by a moving straight line which always touches two curvilinear directrices and remains parallel to a plane directer.

257. To assume a Rectilinear Element of a Warped Surface with Two Curvilinear Directrices and a Plane Directer.

Analysis and Construction. See Fig. 111. Let M–N and O–P represent the two given curvilinear directrices, and let S represent the given plane directer.

In the plane S assume a number of straight lines, as E–B, E–1, E–2, E–3, etc. Through any point A on one of the directrices M–N draw the straight lines A–L, A–K, A–G, A–F, etc., parallel respectively to E–B, E–1, E–2, E–3, etc. These lines through the point A lie in a plane parallel to S.

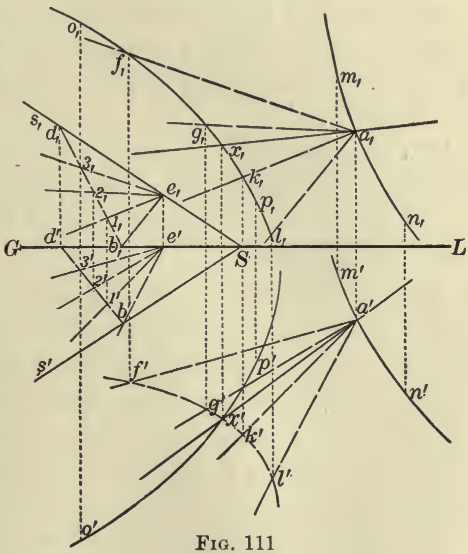

Fig. 111

A–L intersects the horizontal projecting surface of O–P at L; A–K intersects the same surface at K; A–G intersects the same surface at G; and so on. L–K–G–F is the intersection of the plane through the point A with the horizontal projecting surface of O–P. The point X, in which O–P crosses L–K–G–F, is the point in which O–P intersects the plane through the point A. The straight line A–X is the required element, since it is in a plane parallel to S and touches the two directrices M–N and O–P.

The greater the number of auxiliary lines assumed through the point A, the more accurately will the line $L–K–G–F$ be determined.

258. To assume a Point upon a Warped Surface with Two Curvilinear Directrices and a Plane Directer. See Section 255.

259. The Conoid. The conoid is a warped surface with a plane directer and two linear directrices, one rectilinear and the other curvilinear.

If the rectilinear directrix of a conoid is taken perpendicular to the plane directer, the surface is called a *right conoid.*

260. Problem 154. *Assume a rectilinear element upon the warped surface whose rectilinear directrices are* $[A = -6, 3, 1;\ B = 1, 6, 6]$ *and* $[C = 4, -2, 1;\ D = 4, -2, 6]$ *and whose plane directer is* $S = 4, 30°, 20°$.

261. Problem 155. *Assume a rectilinear element upon the warped surface whose rectilinear directrices are* $[A = -4, -4, 2;\ B = -4, 4, 6]$ *and* $[C = 0, 1, 6;\ D = 6, 6, 1]$ *and whose plane directer is* $S = 6, 30°, 60°$.

262. Problem 156. *Assume a rectilinear element upon the warped surface whose rectilinear directrices are* $[A = -6, -3, 4;\ B = 6, -3, 4]$ *and* $[C = -2, -5, 6;\ D = 4, 1, 2]$ *and whose plane directer is* $S = 0, 30°, 60°$.

263. Problem 157. *Given a warped surface with two curvilinear directrices and a plane directer, the latter coincident with V; required to draw a rectilinear element of the surface.*

264. Problem 158. *Given a warped surface with two curvilinear directrices and a plane directer, the latter parallel to G–L but oblique to H and V; required to draw a rectilinear element of the surface.*

265. Problem 159. *Given a warped surface with one rectilinear directrix and one curvilinear directrix and a plane directer; required to draw a rectilinear element of the surface.*

266. Problem 160. *Solve the preceding problem when the rectilinear directrix is perpendicular to H and the plane directer is H.*

267. The Warped Surface with Three Linear Directrices. The warped surface with three linear directrices is generated by a straight line moving in such a way as to touch the three directrices.

Under this class of surfaces one, two, or all the directrices may be rectilinear, or one, two, or all the directrices may be curvilinear.

268. The Hyperboloid of One Nappe. The hyperboloid of one nappe is a warped surface with three rectilinear directrices.

This surface is represented by locating the three rectilinear directrices and a sufficient number of the rectilinear elements, or positions occupied by the generatrix, to reveal the character of that portion of the surface under consideration.

269. To assume a Rectilinear Element of the Hyperboloid of One Nappe. See Fig. 112. Let M–N, O–P, and Q–R represent the three rectilinear directrices.

Assume any point, as A, upon the directrix M–N. Assume any two points, as B and D, upon the directrix O–P. Connect A and B, and also A and D, by straight lines and produce. The plane of these two lines contains the point A and the directrix O–P. By Section 151 find the point G in which the remaining directrix Q–R intersects the plane of the two lines A–B and A–D.

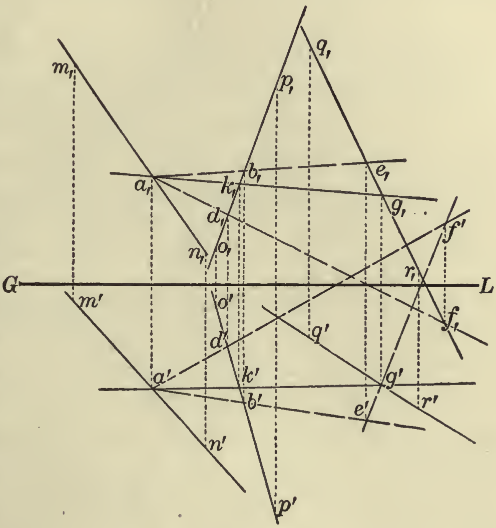

FIG. 112

Connect A and G by a straight line which intersects O–P at K, since A–G and O–P are in the same plane. A–K–G is an element of the surface, since it touches the three directrices.

The two points $k_{\prime}$ and k' should be in the same straight line perpendicular to G–L, which serves as a check upon the work.

270. To assume a Point upon the Surface of the Hyperboloid of One Nappe. First assume an element of the surface and then assume a point upon the element, or follow the directions of Section 255.

271. To assume a Rectilinear Element of a Warped Surface with Three Curvilinear Directrices.

Analysis and Construction. See Fig. 113. Let *M–N*, *O–P*, and *Q–R* represent the three given curvilinear directrices.

Assume a point *A* on one of the directrices *M–N*. Assume a number of points, 1, 2, 3, 4, etc., on one of the other directrices *O–P*. Through *A* and 1, *A* and 2, *A* and 3, etc., draw the straight lines *A*–1–*B*, *A*–2–*D*, *A*–3–*E*, *A*–4–*F*, etc. These lines are elements of a conical surface with vertex at *A*.

A–1–*B* intersects the horizontal projecting surface of *Q–R* at *B*,

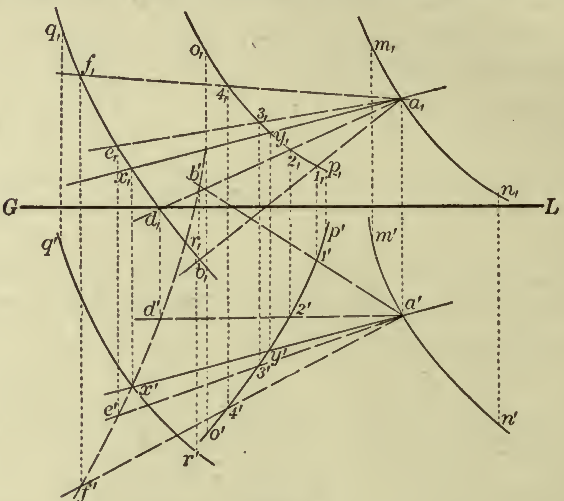

Fig. 113

A–2–*D* intersects the same surface at *D*; *A*–3–*E* intersects the same surface at *E*; and so on.

B–D–E–F is the intersection of the conical surface with the horizontal projecting surface of *Q–R*.

The point *X* in which *B–D–E–F* intersects *Q–R* is the point in which *Q–R* intersects the conical surface.

Connect *A* and *X* by a straight line. This line must intersect *O–P*, since *O–P* lies upon the conical surface of which *A–X* is an element.

A–Y–X is the required element, since it touches the three directrices *M–N*, *O–P*, and *Q–R*.

The points $y_{,}$ and y' should lie in the same straight line perpendicular to $G–L$, and thereby check the work.

The greater the number of auxiliary lines drawn through the point A, the more accurately may the required element be determined.

272. Problem 161. *Assume a rectilinear element upon the hyperboloid of one nappe whose three directrices are* [$A = -6, 6, 2$; $B = -1, -1, 6$], [$C = -2, -2, 1$; $D = 2, 5, 6$], *and* [$E = 0, 6, 1$; $F = 6, 3, 6$].

273. Problem 162. *Assume a rectilinear element upon the hyperboloid of one nappe whose three directrices are* [$A = -6, -2, 6$; $B = -1, 5, 2$], [$C = 0, 2, 2$; $D = 0, 2, 6$], *and* [$E = 2, -2, 1$; $F = 6, 4, 6$].

274. Problem 163. *Given a warped surface with three curvilinear directrices, one in H, another in V, and the third in the third quadrant; required to assume a rectilinear element of the surface.*

275. The Helicoid. The helicoid is a warped surface generated by a straight line moving uniformly around and along a rectilinear directrix which it intersects and with which it makes a constant angle.

276. To represent the Helicoid. The helicoid is represented by locating the directrix or axis (which is usually taken perpendicular to H), a number of the more important rectilinear elements of the surface, and the base or intersection of the surface with H.

To do this we must know the angle which the generatrix makes with the directrix, and the vertical distance through which the generatrix moves for each circuit of the directrix.

In Fig. 114 let $A–B$ represent the directrix, or axis, assumed perpendicular to H. Through any point C ($c_{,}$, c') on the axis draw $C–D$ parallel to V and making the given angle with the axis. Since $C–D$ is taken parallel to V its horizontal projection $c_{,}–d_{,}$ will be parallel to $G–L$, and its vertical projection $c'–d'$ will make the same angle with $a'–b'$ that the generatrix makes with the directrix.

Since the generatrix moves uniformly around and along the directrix, each point of the generatrix will generate a helix whose pitch will be equal to the total rise or fall of the generatrix per revolution, and whose radius will be equal to the distance of the point from the directrix.

Lay off upon $a'-b'$ downward from c' the distance $c'-c^V$ equal to the total fall of the generatrix per revolution, and through c^V draw c^V-d^V parallel to $c'-d'$. The line c^V-d^V is the vertical projection of the generatrix at the end of one circuit.

The point D ($d_{\prime}$, d'), which is the point in which the generatrix in its original position pierces H, will generate a helix whose horizontal projection is the circle $d_{\prime}-d_{\prime\prime}-d_{\prime\prime\prime}-d_{\prime\prime\prime\prime}-\cdots$, and whose vertical projection $d'-d''-d'''-d''''-d^V$ may be found by Section 214.

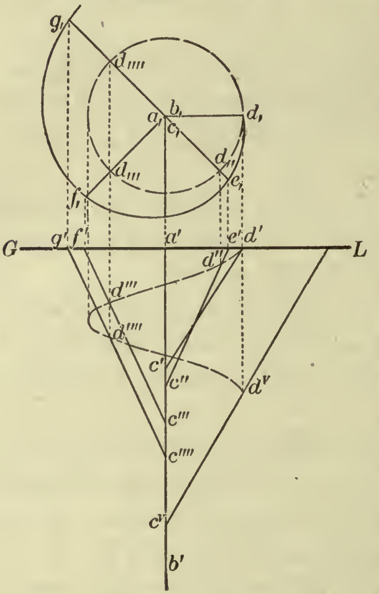

Fig. 114

When the horizontal projection of the point D takes the position $d_{\prime\prime}$, or when the horizontal projection of the generatrix takes the position $c_{\prime}-d_{\prime\prime}$, the vertical projection, d'', of D in this position will fall upon the vertical projection of the helix generated by D and on the straight line through $d_{\prime\prime}$ perpendicular to $G-L$.

This shows that in moving from the first position to this position all points of the generatrix have moved downward a distance equal to the distance of d'' below $G-L$. Therefore, to obtain the vertical projection of C for this position of the directrix, lay off upon $a'-b'$ downward from c' a distance $c'-c''$ equal to the distance of d'' below $G-L$. The point c'' is the vertical projection sought, and the straight line $c''-d''$ is the vertical projection of the generatrix in this position. In the same way we may find the horizontal and vertical projections of the generatrix in other positions, as may be seen from the diagram.

The generatrix in its first position pierces H at $d_{\prime}$, in the second position it pierces H at $e_{\prime}$, in the third position it pierces H at $f_{\prime}$, etc. Since these various positions of the generatrix represent elements of the surface, the curve $d_{\prime}-e_{\prime}-f_{\prime}-g_{\prime}-\cdots$ will represent the base of the surface.

If the angle between the generatrix and the directrix is a right angle, the surface becomes a *right conoid*. The generatrix is a straight line and moves in such a way as to touch two directrices, one rectilinear and the other curvilinear, and remains parallel to a plane directer H, to which the rectilinear directrix is perpendicular.

The helicoid must not be confused with the helical convolute in which the generatrix moves tangentially to the helix and therefore does not intersect the axis.

The helicoid is of practical interest since it has a direct application in the structure of the common screw thread, as may be seen in Sections 282 and 283.

277. To assume a Rectilinear Element on the Surface of the Helicoid. Represent the generatrix in one of its positions as explained above.

278. To assume a Point upon the Surface of the Helicoid. Follow the method explained in Section 255.

279. Problem 164. *Represent the helicoid whose generatrix makes an angle of* 30 *degrees with the axis and whose pitch is* 6 *units.*

280. Problem 165. *Represent the helicoid whose generatrix makes an angle of* 60 *degrees with the axis and whose pitch is* 4 *units.*

281. Problem 166. *Represent the helicoid whose generatrix makes an angle of* 90 *degrees with the axis and whose pitch is* 6 *units.*

282. Problem 167. *To represent a triangular-threaded screw.*

In Fig. 115 let $A–B$ represent the axis of a helicoid, and let $D–A$ and $D–E$ represent two generating elements of the surface, equally inclined to the axis, parallel to V, and intersecting at D.

Lay off upon $D–A$ and $D–E$ from D the equal distances $D–F$ and $D–G$, and through F and G draw the indefinite vertical line $F–K$.

While the generating lines $D–A$ and $D–E$ generate their respective helicoidal surfaces, the three points F, D, and G will generate their respective helices, and the straight line $F–K$ will generate a cylindrical surface whose diameter is equal to $l_1–f_1$.

If in this particular case the pitch of the helicoids is made equal to the distance $F–G$, or some multiple of it, the surface generated by the two sides $D–F$ and $D–G$ of the isosceles triangle $D–F–G$ will be the surface of a triangular-threaded screw.

After a half revolution the two generating lines will take the positions indicated in vertical projection by d''–a'' and d''–e'' respectively, and after a complete revolution they will take the positions indicated in vertical projection by d'''–a''' and d'''–e''' respectively.

We may regard the thread of a triangular-threaded screw as generated by the surface of an isosceles triangle which moves uniformly around and along the surface of a right circular cylinder in such a way that the base of the triangle always rests on the surface of the cylinder, and the plane of the triangle always contains the axis of the cylinder.

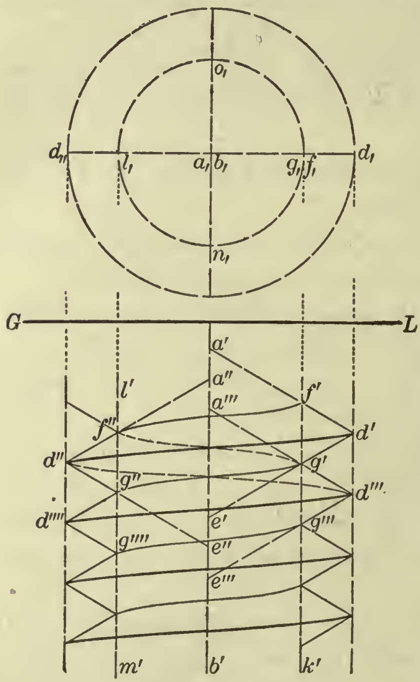

FIG. 115

On account of this motion each apex of the triangle will generate a helix (see Section 213).

If the downward movement of the triangle per revolution is equal to its base, the screw is called single threaded; if equal to twice the base of the triangle, it is called double threaded; and if equal to three times the base of the triangle, it is called triple threaded.

Fig. 115 represents a single-triangular-threaded screw. The cylinder is represented in plan by $f_,$–$n_,$–$l_,$–$o_,$ and in elevation by f'–k'–m'–l'. The generating triangle, when its plane is parallel to V, is represented in plan by $f_,$–$d_,$ and in elevation by f'–d'–g'.

The helices generated by the points F, D, and G may be represented by Section 214, remembering that the pitch in each case is equal to the length of the base of the triangle.

The extreme elements of the thread surface on the right and left are represented in vertical projection by f'–d'–g'–d'''–g'''–$\cdots$ and by f''–d''–g''–d''''–g''''–$\cdots$, which represent the vertical projections

of the sides of the generating triangle when occupying positions parallel to *V*.

283. Problem 168. *To represent a square-threaded screw.*

In Fig. 116 let *A–B* represent the axis of a helicoid, and let *D–E* and *F–G*, equal in length, perpendicular to the axis, parallel to *V*, and at a stated distance apart, represent two generating elements of the surface.

Make *D–K* and *F–L* each equal to *D–F*. Through *D* and *F* draw the indefinite vertical line *D–M*, and through *K* and *L* draw the vertical line *K–N*.

While the generating lines *D–K* and *F–L* generate their respective helicoidal surfaces, the four points *D*, *F*, *L*, and *K* will generate their respective helices, and the two vertical lines *D–M* and *K–N* will generate cylindrical surfaces whose diameters are equal to $d_{\prime}$–$d_{\prime\prime}$ and $k_{\prime}$–$k_{\prime\prime}$ respectively.

If in this particular case the pitch of the helicoidal surfaces is made equal to twice, or four times, or six times, etc., the distance *D–F*, the surface generated by the three sides *K–D*, *D–F*, and *F–L* of the square *K–D–F–L* will be the surface of a square-threaded screw.

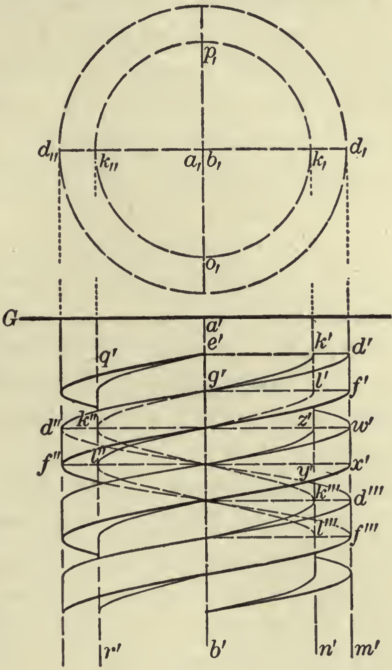

FIG. 116

In Fig. 116 the pitch is made equal to four times the distance *D–F*. After a half revolution the three generating lines *D–K*, *D–F*, and *F–L* will take the positions indicated in vertical projection by d''–k'', d''–f'', and f''–l''; and after a complete revolution they will take the positions indicated in vertical projection by d'''–k''', d'''–f''', and f'''–l'''.

We may regard the thread of a square-threaded screw as generated by the surface of a square which moves uniformly around and

along the surface of a right circular cylinder in such a way that one side of the square always remains on the surface of the cylinder and the plane of the square always contains the axis of the cylinder.

On account of this motion each apex of the square will generate a helix (see Section 213).

If the downward movement of the generating square per revolution is equal to twice the side of the square, the thread is called single threaded, if equal to four times the side of the square it is called double threaded, and if equal to six times the side of the square it is called triple threaded.

Fig. 116 represents a double-square-threaded screw. The inner cylinder is represented in plan by $k_{,}$–$o_{,}$–$k_{,,}$–$p_{,}$ and in elevation by k'–n'–r'–q'. The generating square in its original position, when its plane is parallel to V, is represented in plan by $k_{,}$–$d_{,}$ and in elevation by k'–d'–f'–l'.

The helices generated by the four vertices K, D, F, and L of the square may be represented by Section 214, remembering that the pitch of each helix is equal to four times the side of the generating square.

The thread directly below the one just generated is generated by the square Z–W–X–Y, which moves below the square K–D–F–L and at a distance equal to the side of the square.

The spaces between the threads of a square-threaded screw, so far as the drawing is concerned, may be made equal in dimension to the threads themselves.

CHAPTER XII

REPRESENTATION OF SURFACES OF REVOLUTION

284. General Properties of Surfaces of Revolution. The intersection of a surface of revolution by a plane perpendicular to the axis is the circumference of a circle, since from the nature of the generation of such surfaces (see Section 223) each point of the generatrix generates the circumference of a circle whose plane is perpendicular to the axis.

If two surfaces of revolution have a common axis, and the surfaces either intersect or are tangent to each other, their line of intersection or of tangency will be the circumference of a circle whose center is in the common axis and whose plane is perpendicular to the axis. For if through the axis and any point of the line of intersection or of tangency we pass a plane, it will cut from the two surfaces two lines which will either intersect or be tangent at the assumed point. If now these two lines with their point of intersection or of tangency be revolved about the common axis, the lines will generate their respective surfaces, and the point of intersection or of tangency, which will remain common to the two surfaces, and therefore generate their line of intersection or of tangency, will generate the circumference of a circle whose center is in the axis and whose plane is perpendicular to the axis.

Two surfaces of revolution are tangent to each other when they are tangent to the same surface at a common point, or when planes passed through their point of contact cut from the two surfaces lines which are tangent to each other at the point of contact.

A plane tangent to a single curved surface of revolution at a given point will be tangent to the surface all along the rectilinear element passing through this point (see Section 226).

285. The Meridian Plane and the Meridian Line. Any plane containing the axis of a surface of revolution is called a *meridian plane* and its intersection with the surface is called a *meridian line.*

All meridian lines of the same surface of revolution are the same.

If a plane is tangent to a surface of revolution at a given point, it will be perpendicular to the meridian plane of the surface passing through this point. For if through the point of tangency we pass a plane perpendicular to the axis, it will cut from the surface the circumference of a circle, from the tangent plane a straight line tangent to the circle at the point of tangency, and from the meridian plane a straight line which is the radius of the circle at the point of tangency, and therefore perpendicular to the rectilinear tangent. Through the point of tangency and parallel to the axis draw a straight line. This line is in the meridian plane, is perpendicular to the plane passed perpendicular to the axis, and is therefore perpendicular to the rectilinear tangent.

Since then the tangent plane has a straight line (the rectilinear tangent) perpendicular to two straight lines (the radius and the line parallel to the axis) of the meridian plane, it must be perpendicular to the meridian plane.

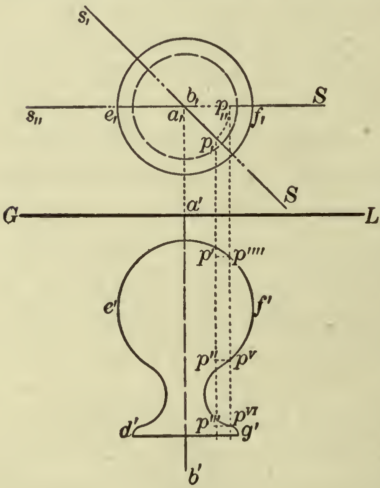

Fig. 117

286. Representation of Surfaces of Revolution. Surfaces of revolution are usually represented by assuming their axes perpendicular to H, although this is not necessary. The intersection of the surface with H, or the horizontal projection of some important section of the surface made by a plane perpendicular to the axis, is taken for the horizontal projection of the surface, and the vertical projection of the meridian curve whose plane is parallel to V is taken as the vertical projection of the surface.

287. To assume a Point upon any Surface of Revolution.

Analysis and Construction. Let the surface of revolution be represented as in Fig. 117, where A–B represents the axis, where the circle $e_{\prime}$–$f_{\prime}$ represents the horizontal projection of the largest horizontal circle of the surface, and where d'–e'–f'–g' represents

the vertical projection of the meridian curve whose plane is parallel to V.

Assume at random the horizontal projection $p_{\prime}$ of the required point. The horizontal projecting line of the point must intersect the given surface in all the points of the surface which can have their horizontal projection at $p_{\prime}$. Through this horizontal projecting line and the axis of the surface pass the meridian plane S, which will cut from the surface a meridian curve. The points in which this meridian curve is intersected by the horizontal projecting line of the point must be the points in question.

Revolve the plane S about A–B as an axis until the plane is parallel to V. The horizontal trace will then take the position S–$s_{\prime\prime}$, the point $p_{\prime}$ will take the position $p_{\prime\prime}$, and the vertical projection of the horizontal projecting line will take the position p''''–p^{V}–p^{VI}. The plane of the meridian curve is now parallel to V and the vertical projection of the meridian curve will be identical with the vertical projection of the surface.

The points p'''', p^{V}, p^{VI}, the points in which the vertical projection of the horizontal projecting line in revolved position intersects the vertical projection of the meridian curve in revolved position, must represent the vertical projections of the required points in revolved position.

After the counter revolution $p_{\prime\prime}$ will take the position $p_{\prime}$, p'''' will take the position p', p^{V} will take the position p'', and p^{VI} will take the position p'''. Therefore any one of the three points p', p'', or p''' may be taken as the vertical projection of the point upon the surface which has its horizontal projection at $p_{\prime}$.

The point may be located by assuming the vertical projection first.

288. Problem 169. *Given a cylinder of revolution whose axis is perpendicular to H; required to represent the cylinder and to assume a point upon the surface.*

289. Problem 170. *Given a cylinder of revolution whose axis is parallel to G–L; required to represent the cylinder and to assume a point upon the surface.*

290. Problem 171. *Given a cone of revolution; required to represent the cone and to assume a point upon the surface.*

291. Problem 172. *Given a sphere whose center is in the third quadrant; required to represent the sphere and to assume a point upon the surface.*

292. Problem 173. *Given a sphere whose center is in G–L; required to represent the sphere and to assume a point upon the surface.*

293. Problem 174. *Given a sphere whose center is in the second quadrant and equidistant from H and V; required to represent the sphere and to assume a point upon the surface.*

294. Problem 175. *Represent an ellipsoid of revolution and assume a point upon the surface.*

295. Problem 176. *Represent a paraboloid of revolution and assume a point upon the surface.*

296. Problem 177. *Represent a hyperboloid of revolution and assume a point upon the surface.*

297. The Hyperboloid of Revolution of One Nappe. The hyperboloid of revolution of one nappe is a warped surface of revolution generated by the revolution of a straight line about a rectilinear axis not in the plane of the generatrix.

To represent the hyperboloid of revolution of one nappe we must know the relation of the generatrix to the axis. This may be expressed by giving the distance of the generatrix from the axis and the inclination of the generatrix to a plane perpendicular to the axis. The distance of the generatrix from the axis will, of course, be measured on a straight line perpendicular to the two (see Section 199).

In Fig. 118 let A–B, assumed perpendicular to H, represent the rectilinear axis.

When the generatrix occupies a position parallel to V, the angle which its vertical projection makes with G–L must equal the angle which the generatrix makes with H; and the perpendicular distance from the point in which the axis pierces H to the horizontal projection of the generatrix must equal the distance of the generatrix from the axis (see Problem 137, Analysis 2).

Therefore through any point, as e', on the vertical projection of the axis draw d'–e'–f', making with G–L the angle which the generatrix is to make with H. Draw $d_{\prime}$–$e_{\prime}$–$g_{\prime}$ parallel to G–L and at

a distance from the point in which the axis pierces H equal to the distance which the generatrix is to be from the axis. D–E–F represents the generatrix in a position parallel to V and between the axis and V.

From the nature of the generation every point of the generatrix will generate the circumference of a circle whose plane is perpendicular to the axis and whose radius is equal to the distance of the point from the axis.

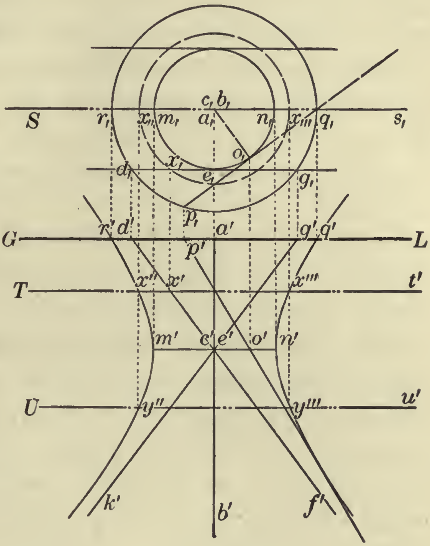

Fig. 118

Since, by Section 199, C–E is the straight line perpendicular to both A–B and D–E–F, the point E of the generatrix is nearer to the axis than any other point of the generatrix, and will generate the circumference of the smallest circle of the surface. This circle is horizontally projected in $m_{,}$–$o_{,}$–$n_{,}$–$m_{,}$ and vertically projected in m'–n', and is called the *circle of the gorge*.

It will be observed that as we increase the distance from E, other points upon the generatrix above or below E will generate circumferences of circles of gradually increasing radii.

It will be also observed that the circumference of the circle generated by a point of the generatrix at any given distance above the plane of the circle of the gorge will have the same radius as the circumference of the circle generated by a point of the generatrix at the same distance below the plane of the circle of the gorge.

This shows that the surface is symmetrical with reference to the plane of the circle of the gorge.

The circumference of the circle $d_{,}$–$p_{,}$–$q_{,}$–$r_{,}$, generated by the point D, the point in which the generatrix D–E–F pierces H, is called the *base* of the surface.

If through E we draw another straight line G–E–K parallel to V and making the same angle with H as D–E–F, and revolve this line about the axis A–B, it will generate a surface identical with the one which we have been considering, since for any given distance above or below the plane of the circle of the gorge points upon the two generatrices are equidistant from the axis.

The surface can then be generated by the revolution of two distinct straight lines. Therefore through any point of the surface two rectilinear elements of the surface may be drawn.

Since every element of the surface intersects the circumference of the circle of the gorge, and since the circle of the gorge is the smallest circle of the surface, the horizontal projection of each element of the surface must contain a point in the horizontal projection of the circumference of the circle of the gorge and yet cannot intersect it. Therefore the horizontal projection of each element of the surface must be tangent to the horizontal projection of the circle of the gorge.

298. To assume a Rectilinear Element of the Hyperboloid of Revolution of One Nappe. See Fig. 118. Assume any point, as P, upon the base. Through $p_{\prime}$ draw $p_{\prime}$–$o_{\prime}$ tangent to the horizontal projection of the circle of the gorge. This is the horizontal projection of an element of the surface piercing H at P and crossing the circle of the gorge at O. P is vertically projected at p' and O is vertically projected at o'. Therefore p'–o' is the vertical projection of the element in question.

299. To assume a Point upon the Surface of the Hyperboloid of Revolution of One Nappe. First assume an element of the surface and then assume a point upon the element.

Or we may assume the horizontal projection of the point at random, through it draw the horizontal projection of the element containing the point, thence the vertical projection of the same element as explained above, and finally the vertical projection of the point upon the vertical projection of the element.

300. To determine the Meridian Curve of the Hyperboloid of Revolution of One Nappe. We shall consider the section of the surface made by the meridian plane parallel to V, for the vertical projection of this curve will be equal in every respect to the curve

itself. In Fig. 118, S–$s_{/}$ is the horizontal trace of a meridian plane parallel to V. This plane intersects the circumference of the circle of the gorge at M and N. Therefore m' and n' are two points in the vertical projection of the required meridian curve.

For the same reason r' and q' are two more points in the vertical projection of the curve, since R and Q are the two points in which this same meridian plane intersects the base of the surface.

To find other points of the curve, draw horizontal planes, for example T. This plane cuts the generatrix D–E–F at X ($x_{/}$, x') and cuts the surface itself in the circumference of a circle whose horizontal projection is $x_{/}$–$x_{///}$–$x_{//}$ and whose vertical projection is x''–x'''. This circumference intersects the meridian plane at X ($x_{//}$, x''), also at X ($x_{///}$, x'''), locating two points in the required curve.

Since the surface is symmetrical with reference to the plane of the circle of the gorge, the meridian curve in vertical projection will be symmetrical with reference to the vertical projection of the circle of the gorge. Therefore, having determined the points x'' and x''' in the vertical projection of the curve, corresponding points y'' and y''' below m'–n' may be obtained by drawing U–u' parallel to m'–n' and at the same distance below m'–n' as T–t' is above, and by taking y'' and y''' at the same distance from a'–b' as x'' and x'''. By this process we may find as many points in the curve as may be desired.

CHAPTER XIII

DETERMINATION OF PLANES TANGENT TO SURFACES OF SINGLE CURVATURE

301. General Instructions. Note carefully the conditions imposed by the problem; then by an application of the principles already established with reference to tangency, determine at least two straight lines of the required plane. The points in which these lines intersect H and V will be points in the horizontal and vertical traces of the required plane.

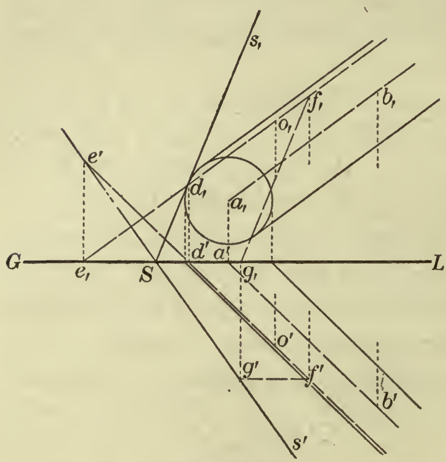

Fig. 119

302. Problem 178. *To draw a plane tangent to a cylinder at a point on the surface.*

Analysis. Since the surface is of single curvature, the rectilinear element of the surface through the assumed point is the element of tangency and must be a line of the required plane (see Section 226).

Since the required plane will be tangent to the surface all along this element of tangency, a straight line tangent to the surface at any point on this element will be a line of the required plane (see Section 226).

Case 1. *When the base of the cylinder is a circle on H.*

Construction. In Fig. 119 let $A–B$ represent the axis of the cylinder, and let O, assumed as in Section 234, represent the point on the surface.

According to analysis the element $O–D$ through O is a line of the required plane. This element pierces H at $d_{,}$ and pierces V at

e', the former a point in the horizontal trace and the latter a point in the vertical trace of the required plane.

Through $d_{\prime}$ draw S–$d_{\prime}$–$s_{\prime}$ tangent to the circular base at $d_{\prime}$. This is a line of the required plane, since it is tangent to a curve of the surface at a point on O–D. As this tangent is also in H it is the required horizontal trace. To draw the required vertical trace, connect S with e' and produce.

Check. Through F, any point on the element O–D, draw F–G parallel to $s_{\prime}$–S. This must be a line of the required plane, since it is drawn through a point F in the plane parallel to a line $s_{\prime}$–S also in the plane, and therefore should pierce V in the vertical trace already found.

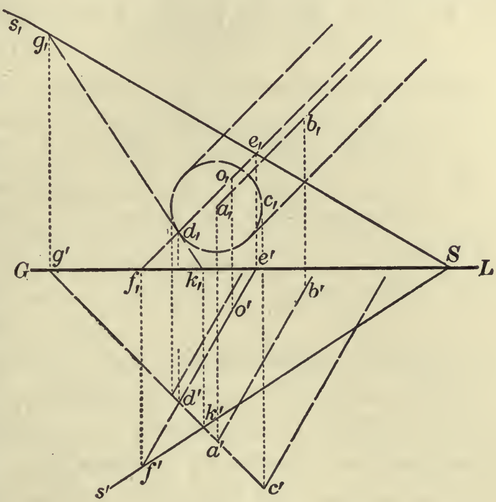

FIG. 120

CASE 2. *When the plane of the base of the cylinder is oblique to H but perpendicular to V.*

Construction. In Fig. 120 let A–B represent the axis of the cylinder, let the circle with center at $a_{\prime}$ represent the horizontal projection of the base, let the straight line d'–a'–c' represent the vertical projection of the base, and let O represent the point on the surface.

The element of tangency O–D through O pierces H at $e_{\prime}$ and pierces V at f', points in the required traces.

Through $d_{\prime}$ draw $d_{\prime}$–$g_{\prime}$ tangent to the horizontal projection of the base. The line $d_{\prime}$–$g_{\prime}$ is the horizontal projection, and the line d'–a'–c' is the vertical projection of a line tangent to the base of the cylinder at the point D on the element O–D, and therefore a line of the required plane. D–G pierces H at $g_{\prime}$ and pierces V at k', two more points in the required traces. The two traces S–$s_{\prime}$ and S–s' are now located.

Check. Same as in Case 1.

CASE 3. *When the cylinder is one of revolution with its axis parallel to G–L.*

Construction. Let the cylinder be represented as in Fig. 121. In assuming the point upon the surface, take the horizontal projection $o_,$ at random. Through this point draw a profile plane cutting the surface of the cylinder in the circumference of a circle whose center is C. The point in which this circumference is intersected by the horizontal projecting line of O must be the point upon the surface. Revolve the profile plane about its horizontal trace as an axis into H. The circle C will take the position of the circle whose center is c_H, and the horizontal projecting line will take the position $o_,–o_H'$. Therefore either o_H or k_H may be taken as the revolved position of the point upon the surface. Let us take the point O, which in this revolved position is vertically projected at o_H' but in true position is vertically projected at o'.

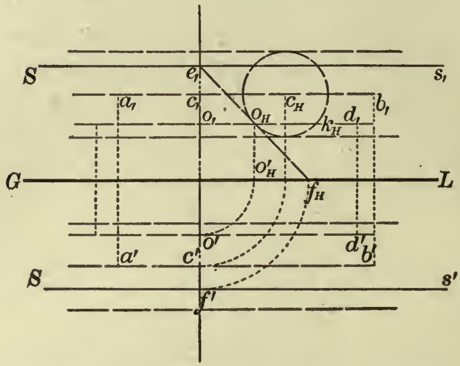

FIG. 121

The element $O–D$ through O is a line of the required plane, and since this line is parallel to $G–L$, we know that the traces of the required plane will both be parallel to $G–L$.

Through o_H draw $e_,–o_H–f_H$ tangent at o_H to the circle whose center is c_H.

This line is the revolved position of a line tangent to a curve of the surface at a point on $O–D$, and therefore is the revolved position of a line of the required plane. The line $E–F$ in true position pierces H at $e_,$, a point in the required horizontal trace, and pierces V in f', a point in the required vertical trace. The two traces $S–s_,$ and $S–s'$ are now located.

Check. Pass another profile plane cutting from the surface of the cylinder the circumference of a circle, and from $O–D$ a point

upon this circumference. The rectilinear tangent to the circle at this point is a line of the required plane and should pierce H and V in the traces already found.

303. Problem 179. *Given a cylinder whose base is a circle on V and whose axis is oblique to H and V; required to assume a point upon the surface and to draw a plane tangent to the surface at this point.*

304. Problem 180. *Given a cylinder whose axis is oblique to H and V, the plane of whose base is perpendicular to H but oblique to V, and the vertical projection of whose base is a circle; required to assume a point upon the surface and to draw a plane tangent to the surface at this point.*

305. Problem 181. *Given a cylinder whose axis is* [$A = 0, 4, 0$; $B = 0, 8, 6$] *and whose base on H is a circle of 3-unit radius; required to assume a point upon the surface and to draw a plane tangent to the surface at this point.*

306. Problem 182. *Given a cylinder whose axis is oblique to H and V and whose base is a circle in a plane parallel to H and above it; required to assume a point upon the surface and to draw a plane tangent to the surface at this point.*

307. Problem 183. *To draw a plane tangent to a cylinder and through a point without the surface.*

Analysis. The required plane will contain an element of the surface of the cylinder. Therefore a straight line through the given point and parallel to the elements of the cylinder will be a line of the required plane (see Section 42) and should pierce H and V in points of the required traces.

Since the required plane will be tangent to the surface all along the element of tangency, any plane oblique to the elements of the cylinder will cut from the surface a curved line, from the auxiliary line parallel to the elements a point, and from the required plane a straight line passing through this point and tangent to the curve at a point on the element of tangency. Therefore, if through any point of the auxiliary line we draw a plane cutting from the surface of the cylinder a curved line, the straight line through the assumed point and tangent to the curve will be a line of the required plane.

CASE 1. *When the base of the cylinder is a circle on H.*

Construction. In Fig. 122 let A–B represent the axis of the cylinder, and let O represent the point assumed without the surface.

Through O draw O–D parallel to the elements of the cylinder and produce it to pierce H at $d_{,}$. Through $d_{,}$ draw $s_{,}$–$d_{,}$–$e_{,}$–S tangent to the circular base at $e_{,}$. According to analysis this is a line of the required plane, and since it is in H, it is the required horizontal trace. Through O draw O–G parallel to $s_{,}$–S piercing V at g'. S–g'–s' is the required vertical trace.

The element E–F through E is the element of tangency.

Since from the point $d_{,}$ another straight line may be drawn tangent to the base of the cylinder, another plane answering the conditions of the problem may be constructed.

Check. Note whether the element of tangency, which is a line of the required plane, pierces V in the vertical trace already located.

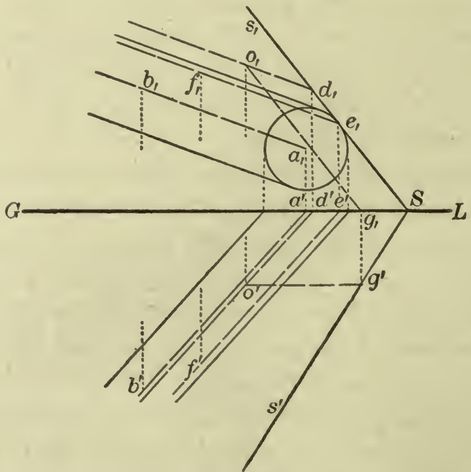

FIG. 122

CASE 2. *When the plane of the base of the cylinder is perpendicular to H but oblique to V.*

Construction. In Fig. 123 let A–B represent the axis of the cylinder, and let O represent the point without the surface.

Through O draw O–D parallel to the elements of the cylinder and produce it to pierce H at $d_{,}$ and to pierce V at e'.

Through F, the point in which $O-D$ pierces the plane of the base of the cylinder, draw a rectilinear tangent to the curve of the base. This tangent will be vertically projected in $f'-g'$, tangent to the vertical projection of the base, and horizontally projected in $f_{,}-g_{,}-a_{,}$. $F-G$ is by analysis a line of the required plane and pierces H at $l_{,}$ and pierces V at m'. $S-s_{,}$ drawn through $d_{,}$ and $l_{,}$

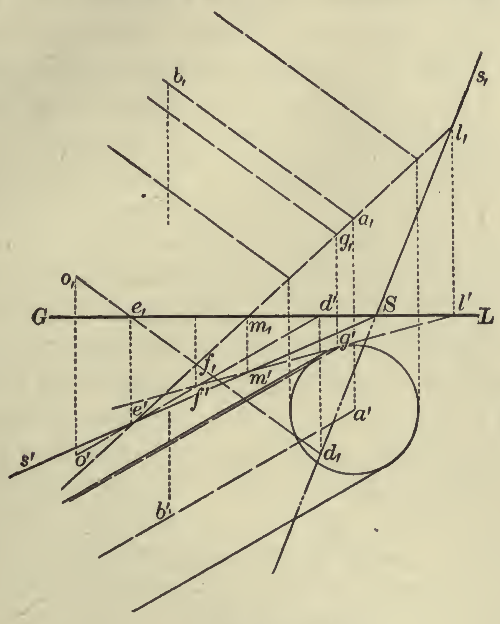

Fig. 123

is the required horizontal trace, and $S-s'$ drawn through e' and m' is the required vertical trace.

The element of tangency is that element determined by the point G.

Since another straight line may be drawn through F tangent to the curve of the base, another plane answering the conditions of the problem may be constructed.

Check. Note whether the element of tangency, which is a line of the required plane, pierces H and V in the traces now located.

CASE 3. *When the cylinder is one of revolution with its axis parallel to G–L.*

Construction. In Fig. 124 let A–B represent the axis of the cylinder, and let O represent the point without the surface.

A straight line through O parallel to the axis A–B will be a line of the required plane, and since this line is parallel to G–L, we know that both traces of the required plane will be parallel to G–L.

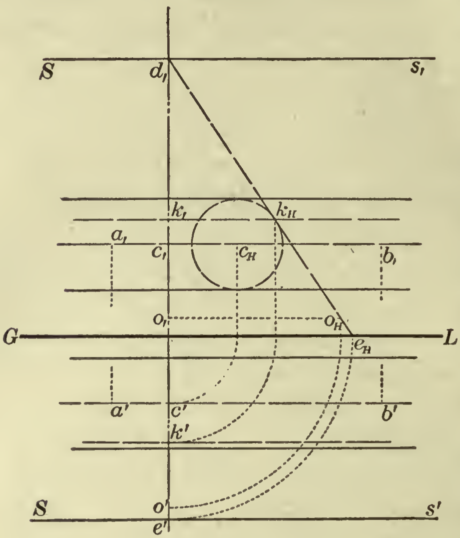

FIG. 124

Through O pass a profile plane cutting the cylinder in the circle whose center is C. Revolve this profile plane about its horizontal trace into H. O will fall at o_{H} and the circle C will take the position of the circle whose center is c_{H}. Through o_{H} draw o_{H}–$d_{\prime}$ tangent to the circle c_{H}. According to analysis this is the revolved position of a line of the required plane. In true position this line pierces H at $d_{\prime}$, a point in the required horizontal trace; and pierces V at e', a point in the required vertical trace. The two traces S–$s_{\prime}$ and S–s' are now located.

The element of tangency is that one determined by the point K.

Since through o_{H} another straight line may be drawn tangent to the circle c_{H}, another plane answering the conditions of the problem may be constructed.

Check. Through some point of the element of tangency draw a profile plane cutting from the cylinder a circle. A rectilinear tangent to this circle at the point assumed is a line of the required plane and should pierce H and V in the traces already located.

308. Problem 184. *Given a cylinder in the third quadrant whose axis is oblique to H and V and whose base is a circle on V; required to assume a point without the surface and to draw through this point a plane tangent to the cylinder.*

309. Problem 185. *Given a cylinder whose axis is oblique to H and V, the plane of whose base is perpendicular to V but oblique to H, and the horizontal projection of whose base is a circle; required to assume a point without the surface and to draw a plane through this point tangent to the cylinder.*

310. Problem 186. *Given a cylinder whose axis is oblique to H and V and whose base is a circle in a plane parallel to V and in front of it; required to assume a point without the surface and to draw through this point a plane tangent to the cylinder.*

311. Problem 187. *Given a cylinder whose axis is* $[A = -4, 0, -4;\ B = -4, 6, 4]$ *and whose base on V is a circle of* 3*-unit radius; also given the point* $O = 2, 4, 2$; *required to draw through the point a plane tangent to the cylinder.*

312. Problem 188. *To draw a plane tangent to a cylinder and parallel to a given straight line.*

Analysis. Since the required plane must contain an element of the cylinder and must be parallel to the given line, the required plane will be parallel to any plane which is parallel both to the axis of the cylinder and to the given line. Therefore, through the given line pass a plane parallel to the axis of the cylinder and draw the required plane tangent to the cylinder and parallel to this plane.

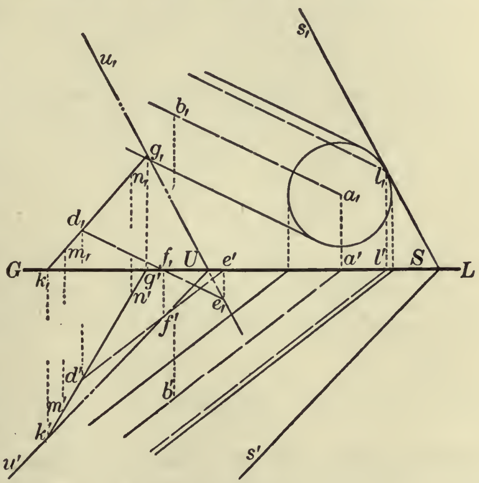

Fig. 125

Case 1. *When the base of the cylinder is a circle on H.*

Construction. In Fig. 125 let A–B represent the axis of the cylinder and let M–N represent the given line.

Through any point D of M–N draw D–E parallel to A–B. D–E pierces H at e_1 and pierces V at f'; M–N pierces H at g_1 and pierces V at k'. The plane U whose traces are located by the

points just found will, by analysis, be parallel to the required plane. The horizontal trace of the required plane, then, must be parallel to U–$u_{\prime}$ and must also be tangent to the circular base on H whose center is $a_{\prime}$. Therefore draw the horizontal trace S–$s_{\prime}$ under these conditions, and through S draw the vertical trace S–s' parallel to U–u'.

The element of tangency may be located by drawing through L, the point of tangency on the base, a straight line parallel to the axis.

Since another straight line can be drawn parallel to U–$u_{\prime}$ and tangent to the circle whose center is $a_{\prime}$, another plane answering the conditions of the problem may be constructed.

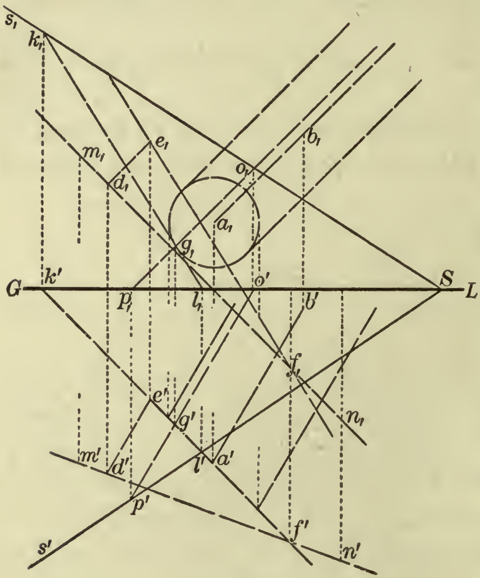

Fig. 126

Check. Through some point on the element of tangency draw a straight line parallel to M–N and note whether it pierces H and V in the traces already located.

Case 2. *When the plane of the base of the cylinder is perpendicular to V but oblique to H.*

Construction. In Fig. 126 let A–B represent the axis of the cylinder and let M–N represent the given line.

Through any point D of the given line M–N draw D–E parallel to the axis A–B. The plane of the two lines M–N and D–E is parallel to the required plane. D–E pierces the plane of the base of the cylinder at E, and M–N pierces the same plane at F. The line E–F is the intersection of the auxiliary plane with the plane of the base of the cylinder, and must be parallel to the intersection of the required plane with the plane of the base.

The intersection of the required plane with the plane of the base must also be tangent to the curve of the base. Therefore $k_{\prime}$–$g_{\prime}$–$l_{\prime}$, drawn parallel to $e_{\prime}$–$f_{\prime}$ and tangent to the circle whose center is $a_{\prime}$, is the horizontal projection, and k'–g'–l' is the vertical projection of the intersection of the required plane with the plane of the base. K–G–L, which is a line of the required plane, intersects H at $k_{\prime}$ and intersects V at l', points respectively in the horizontal and vertical traces of the required plane.

Through G draw the element of tangency G–O, another line of the required plane, and produce it to pierce H at $o_{\prime}$ and to pierce V at p'. The two required traces S–$o_{\prime}$–$k_{\prime}$–$s_{\prime}$ and S–l'–p'–s' may now be drawn.

Since another straight line can be drawn parallel to $e_{\prime}$–$f_{\prime}$ and tangent to the circle whose center is $a_{\prime}$, another plane answering the conditions of the problem may be constructed.

Check. Through some point of the element of tangency draw a straight line parallel to the given line M–N, and note whether it intersects H and V in the traces now located.

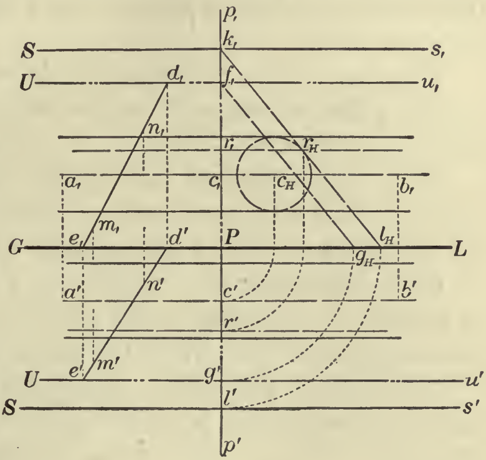

FIG. 127

CASE 3. *When the cylinder is one of revolution with its axis parallel to G–L.*

Construction. In Fig. 127 let A–B represent the axis of the cylinder and let M–N represent the given line.

Since the axis of the cylinder is parallel to G–L, the traces of the required plane will also be parallel to G–L.

Through M–N draw the plane U parallel to the axis A–B. This auxiliary plane must be parallel to the required plane.

Draw a profile plane P cutting the cylinder in the circle whose center is C, and cutting the auxiliary plane U in the line F–G. The intersection of the required plane by the profile

plane must be parallel to $F–G$ and be tangent to the circle C, since the required plane is both parallel to the auxiliary plane and tangent to the surface of the cylinder (see Section 226).

Revolve the profile plane about its horizontal trace into H. The circle C falls at the circle whose center is c_H, and the line $F–G$ falls at $f_{,}–g_H$.

The line $k_{,}–l_H$ drawn parallel to $f_{,}–g_H$ and tangent to the circle c_H is the revolved position of the intersection of the required plane with the profile plane. In true position $K–L$ pierces H at $k_{,}$ and pierces V at l', points respectively in the horizontal and vertical traces of the required plane. The two traces $S–s_{,}$ and $S–s'$ may now be drawn.

The element of tangency may be determined by drawing through R, the point of tangency on the circle C, a straight line parallel to the axis $A–B$.

Since another straight line tangent to the circle c_H and parallel to $f_{,}–g_H$ may be drawn, another plane answering the conditions of the problem may be constructed.

Check. Through some point on the element of tangency draw a straight line parallel to the given line $M–N$, and note whether this line pierces H and V in the traces now located.

313. Problem 189. *Given a cylinder whose axis is oblique to H and V and whose base is a circle on V; required to assume a straight line and to draw a plane tangent to the cylinder and parallel to the line.*

314. Problem 190. *Given a cylinder whose axis is oblique to H and V, the plane of whose base is perpendicular to H but oblique to V, and the vertical projection of whose base is a circle; required to assume a straight line and to draw a plane tangent to the cylinder and parallel to the line.*

315. Problem 191. *Given a cylinder whose axis is parallel to G–L; required to assume a straight line in a profile plane and to draw a plane tangent to the cylinder and parallel to the line.*

316. Problem 192. *Given a cylinder whose axis is oblique to H and V and whose base is a circle in a plane parallel to H and above it; required to assume a straight line parallel to G–L and to draw a plane tangent to the cylinder and parallel to the line.*

317. Problem 193. *To draw a plane tangent to a cone at a point on the surface.*

Analysis. Since the surface of a cone is of single curvature, the rectilinear element of the surface through the point of tangency is the element of tangency, and is therefore a line of the required plane (see Section 226).

Since the required plane will be tangent to the surface all along this element, a line tangent to the surface at any point on this element will be a line of the required plane.

CASE 1. *When the base of the cone is a circle on H.*

Construction. In Fig. 128 let $A–B$ represent the axis of the cone, and let O, assumed as in Section 242, represent the point on the surface.

The element $B–O–D$, which is a line of the required plane, pierces H at $d_{\prime}$ and pierces V at e', points in the required traces. Through

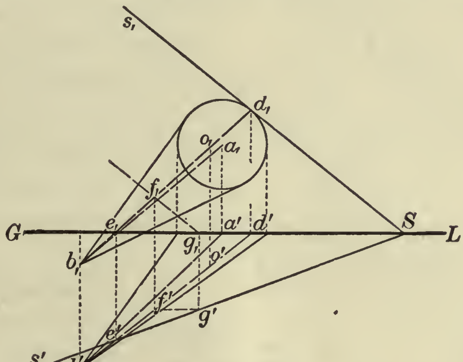

FIG. 128

$d_{\prime}$ draw $S–d_{\prime}–s_{\prime}$ tangent to the circular base at $d_{\prime}$. This is a line of the required plane, because it is tangent to a curve of the surface at a point on $O–D$. Since this tangent is also in H, it is the required horizontal trace. Connect S and e' by a straight line and produce it to represent the required vertical trace.

Check. Through F, any point on the element of tangency $O–D$, draw $F–G$ parallel to $s_{\prime}–S$. This line must be a line of the required plane, since it is drawn through a point F in this plane and parallel to a line $s_{\prime}–S$ also in this plane, and should therefore pierce V in the vertical trace already found.

CASE 2. *When the plane of the base of the cone is perpendicular to V but oblique to H.*

Construction. In Fig. 129 let A–B represent the axis of the cone, let the circle whose center is $a_{,}$ represent the horizontal projection of the base, let the straight line g'–k'–a'–d' represent the vertical projection of the plane of the base, and let O represent the point on the surface.

The element B–O–D through O pierces H at $e_{,}$ and pierces V at f', points in the required traces.

Through $d_{,}$ draw $d_{,}$–$g_{,}$ tangent to the horizontal projection of the base. The line $d_{,}$–$g_{,}$ is the horizontal projection and d'–g' is the vertical projection of a line tangent to the base of the cone at a point D on the element B–O–D, and therefore a line of the required plane. D–G pierces H at $g_{,}$ and pierces V at k', two more points in the required traces.

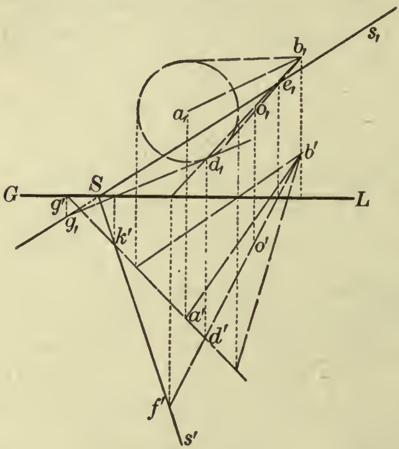

FIG. 129

Check. Same as in Case 1.

318. Problem 194. *Given a cone whose axis is oblique to H and V and whose base is a circle on V; required to draw a plane tangent to the cone at a point on the surface.*

319. Problem 195. *Given a cone whose axis is oblique to H and V and whose base is in a plane perpendicular to H but oblique to V; required to draw a plane tangent to the cone at a point on the surface.*

320. Problem 196. *Given a cone whose axis is oblique to H and V and whose base is a circle in a plane parallel to H and above it; required to draw a plane tangent to the cone at a point on the surface.*

321. Problem 197. *To draw a plane tangent to a cone and through a point without the surface.*

Analysis. Since the required plane must contain the vertex of the cone, a straight line through the vertex of the cone and the

given point must be a line of the required plane. The remainder of the analysis differs little from that given in connection with Problem 183, Section 307.

CASE 1. *When the base of the cone is a circle on H.*

Construction. In Fig. 130 let $A–B$ represent the axis of the cone, and let O represent the point without the surface.

Connect O and B by a straight line and produce it to pierce H at $d_{\prime}$ and to pierce V at e'. Through $d_{\prime}$ draw $d_{\prime}–S–f_{\prime}–s_{\prime}$ tangent to the circular base at $f_{\prime}$.

According to analysis this is a line of the required plane, and since it is in H, it is the required horizontal trace. Through S and e' draw the required vertical trace $S–s'$.

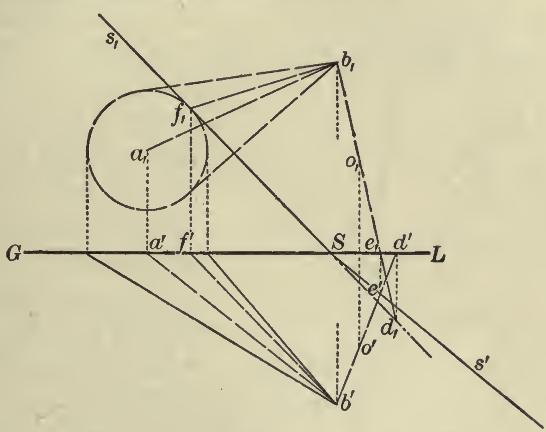

FIG. 130

The element $B–F$ is the element of tangency and a line of the required plane.

Since from the point $d_{\prime}$ another straight line may be drawn tangent to the base, another plane answering the conditions of the problem may be constructed.

Check. Note whether the element of tangency pierces V in the vertical trace now located.

CASE 2. *When the plane of the base of the cone is perpendicular to H but oblique to V.*

Construction. In Fig. 131 let $A–B$ represent the axis of the cone and let O represent the point without the surface.

Connect the vertex of the cone and O by a straight line and produce it to pierce H in $d_{\prime}$ and to pierce V in e', points in the required traces.

Through F, the point in which B–O intersects the plane of the base of the cone, draw a rectilinear tangent to the curve of the base. This tangent will be vertically projected in f'–g', tangent to the vertical projection of the base, and will be horizontally projected in $f_{\prime}$–$g_{\prime}$–$a_{\prime}$.

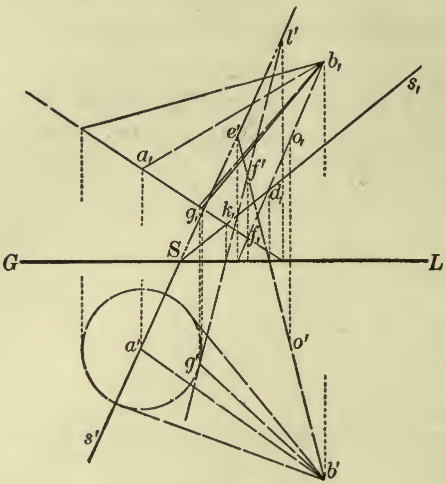

Fig. 131

By analysis F–G is a line of the required plane and pierces H at $k_{\prime}$ and pierces V at l'. S–$s_{\prime}$ drawn through $d_{\prime}$ and $k_{\prime}$ is the required horizontal trace, and S–s' drawn through e' and l' is the required vertical trace.

The element of tangency is the straight line through G and the vertex of the cone.

Since another straight line may be drawn through F tangent to the curve of the base, another plane answering the conditions of the problem may be constructed.

Check. Note whether the element of tangency, which is a line of the required plane, pierces H and V in the traces now located.

322. Problem 198. *Given a cone whose axis is oblique to H and V and whose base is a circle on V; required to assume a point without the surface and to draw through this point a plane tangent to the cone.*

323. Probem 199. *Given a cone whose axis is oblique to H and V, the plane of whose base is perpendicular to V but oblique to H, and the horizontal projection of whose base is a circle; required to assume a point without the surface and to draw a plane through this point and tangent to the cone.*

324. Problem 200. *Given a cone whose axis is oblique to H and V, and whose base is a circle in a plane parallel to H and above it;*

required to assume a point without the surface and to draw a plane through this point and tangent to the cone.

325. Problem 201. *To draw a plane tangent to a cone and parallel to a given straight line.*

Analysis. Since the required plane must contain the vertex of the cone, a straight line through the vertex of the cone and parallel to the given line will be a line of the required plane. A plane containing this auxiliary line and tangent to the cone will be the required plane.

CASE 1. *When the base of the cone is a circle on H.*

Construction. In Fig. 132 let A–B represent the axis of the cone and let M–N represent the given line.

Through the vertex of the cone draw B–D parallel to M–N, piercing H at $d_{,}$ and piercing V at e'. Through $d_{,}$ and tangent to

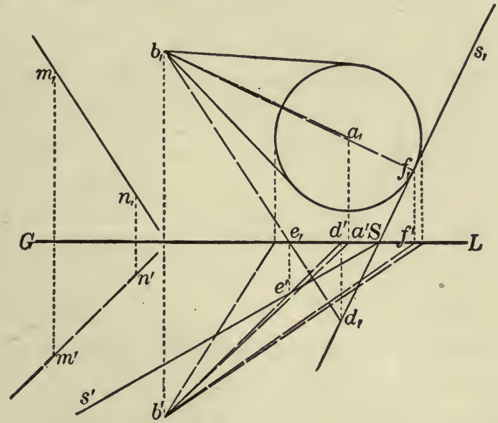

FIG. 132

the circular base draw the required horizontal trace $d_{,}$–S–$s_{,}$. Through S and e' draw the required vertical trace S–e'–s'.

The element of tangency is the straight line drawn through the vertex of the cone and the point of tangency F on the base.

Since another straight line may be drawn through $d_{,}$ and tangent to the circular base, another plane answering the conditions of the problem may be constructed.

Check. Note whether the element of tangency pierces V in the vertical trace already located.

Case 2. *When the plane of the base of the cone is perpendicular to V but oblique to H.*

Construction. In Fig. 133 let A–B represent the axis of the cone and let M–N represent the given line.

Through the vertex of the cone draw E–B–D parallel to M–N, piercing H at $d_{,}$ and piercing V at e', points in the required traces. Through F, the point in which B–D intersects the plane of the base of the cone, draw G–F–O tangent to the curve of the base. The horizontal projection of this tangent is $g_{,}$–$f_{,}$–$o_{,}$, and the vertical projection of this tangent is g'–f'–o'. By analysis G–F–O is a line of the required plane, and pierces H at $g_{,}$ and pierces V at k'. Through $g_{,}$ and $d_{,}$ draw the required horizontal trace S–$s_{,}$. Through k' and e' draw the required vertical trace S–s'.

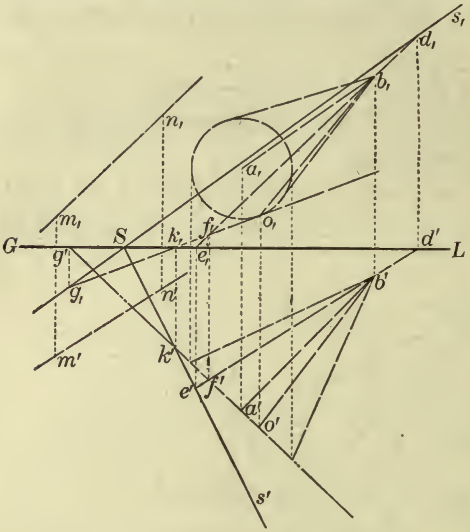

Fig. 133

The element of tangency is the straight line drawn through the point of tangency on the base and the vertex of the cone.

Since another straight line may be drawn through F and tangent to the curve of the base, another plane answering the conditions of the problem may be constructed.

Check. Note whether the element of tangency intersects H and V in the traces now located.

326. Problem 202. *Given a cone whose axis is oblique to H and V and whose base is a circle on V; required to assume a straight line in a profile plane and to draw a plane tangent to the cone and parallel to the line.*

327. Problem 203. *Given a cone whose axis is oblique to H and V, the plane of whose base is perpendicular to H but oblique to V, and the vertical projection of whose base is a circle; required to assume*

a straight line and to draw a plane tangent to the cone and parallel to the line.

328. Problem 204. *Given a cone whose axis is oblique to H and V and whose base is a circle in a plane parallel to V and in front of it; required to assume a straight line and to draw a plane tangent to the cone and parallel to the line.*

329. Problem 205. *To draw a plane tangent to a helical convolute at a point on the surface.*

Analysis. Since the surface of the helical convolute is of single curvature, the rectilinear element of the surface through the point of tangency will be the element of tangency, and therefore be a line of the required plane (see Section 226).

Since the required plane will be tangent to the surface all along the element of tangency, a straight line tangent to the surface at any point of this element of tangency will be a line of the required plane.

Construction. Let the helical convolute be represented as in Fig. 134. *A–B* represents the axis, *D–E–F–G–K* represents the helical directrix, $d_{,}$–$l_{,}$–$m_{,}$–$n_{,}\cdots$ represents the base of the surface, and *O*, assumed as in Section 250, represents the given point on the surface.

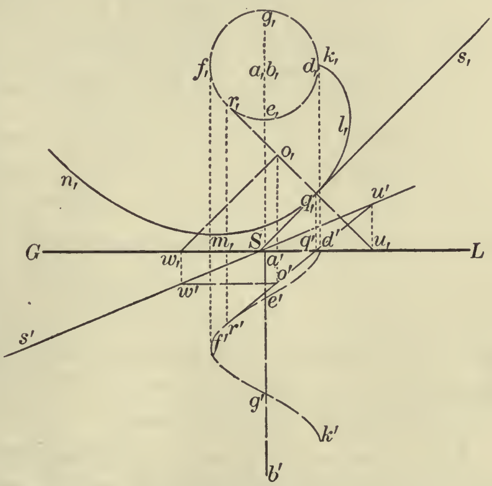

Fig. 134

The element *R–Q–U* through *O* is a line of the required plane and pierces *H* at $q_{,}$ and pierces *V* at u', points in the required traces.

Through $q_{,}$ draw *S*–$q_{,}$–$s_{,}$ tangent to the base of the convolute at $q_{,}$. This is a line of the plane, since it is tangent to a curve of the surface at a point on *O–Q* (see Section 226).

Since this tangent is also in *H*, it is the required horizontal trace. The straight line through *S* and u' is the required vertical trace.

The element $R–Q–U$ is the element of tangency.

Check. Through O, or through any point on the element of tangency, draw a straight line parallel to $S–s_{,}$ and note whether it intersects V in the vertical trace already located.

330. Problem 206. *To draw a plane tangent to a helical convolute and through a point assumed without the surface.*

Analysis. A plane through the given point and intersecting the surface will cut from the surface a line, usually curved, and from the required plane a straight line running through the given point and tangent to the curve. Therefore, through the given point pass a plane cutting from the surface a curved line. Through the given point and tangent to this curved line draw a straight line. This straight line is a line of the required plane and will pierce H and V in points of the required traces.

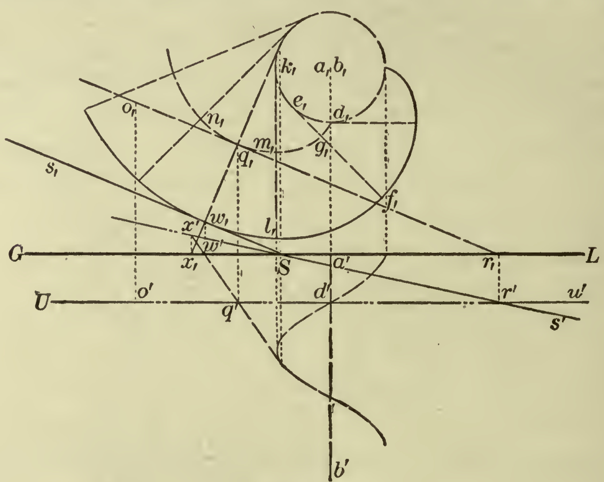

Fig. 135

The element of the surface through the point of tangency just determined on the curve of the surface is the element of tangency and is a line of the required plane.

A straight line tangent to the surface at any point on the element of tangency is another line of the required plane.

Construction. Let the helical convolute be represented as in Fig. 135. $A–B$ represents the axis and O represents the given point.

Through O pass the plane U parallel to H. The plane U cuts the helical directrix at the point D, and since the plane is parallel to H, it cuts the surface of the convolute in a curve similar to the curve cut from the same surface by the plane H, but starting from the point D.

The plane U cuts the element whose horizontal projection is $e_{,}$–$f_{,}$ at a point horizontally projected at $g_{,}$, where $e_{,}$–$g_{,}$ is equal to the rectified arc $e_{,}$–$d_{,}$ (see Section 216).

The plane U cuts the element whose horizontal projection is $k_{,}$–$l_{,}$ at a point horizontally projected at $m_{,}$, where $k_{,}$–$m_{,}$ is equal to the rectified arc $k_{,}$–$e_{,}$–$d_{,}$.

In the same way other points in which the plane U cuts the elements of the surface may be found. The curve $d_{,}$–$g_{,}$–$m_{,}$–$n_{,} \cdots$ is then the horizontal projection of the curve cut from the helical convolute by the plane U.

Through $o_{,}$ draw $o_{,}$–$q_{,}$ tangent to the curve $d_{,}$–$g_{,}$–$m_{,}$–$n_{,}$ at the point $q_{,}$. The line $o_{,}$–$q_{,}$ is the horizontal projection and the line o'–q' is the vertical projection of a straight line through O and tangent to the curve cut from the surface by the plane U. O–Q is then a line of the required plane and pierces V in r', a point in the required vertical trace. Through Q and tangent to the helical directrix draw the element of tangency Q–W–X, piercing H at $w_{,}$ and piercing V at x', points in the required traces. Through $w_{,}$ draw the required horizontal trace S–$s_{,}$ tangent to the base of the surface at $w_{,}$. Through S and x' draw the required vertical trace S–s', which should also pass through r', and check the work.

Check. Through any point of the element of tangency draw a straight line parallel to S–$s_{,}$ and note whether it pierces V in the vertical trace now located.

331. Problem 207. *To draw a plane tangent to a helical convolute and parallel to a given straight line.*

Analysis. Assuming the axis of the convolute perpendicular to H, with some point on the given line as a vertex construct a cone whose elements shall make with H the constant angle which the elements of the convolute make with H.

Each element of the convolute will then have a corresponding element on the cone to which it will be parallel; and any two

planes, the one tangent to the cone and the other tangent to the convolute, along such corresponding parallel elements, will be parallel planes.

Through the given line pass a plane tangent to this auxiliary cone. Such a plane will be parallel to the required plane, since it contains the given line and an element on the cone which is parallel to the element of tangency on the convolute.

Construction. Let the helical convolute be represented as in Fig. 136. A–B represents the axis and M–N represents the given line.

Through n', the point in which M–N pierces V, draw n'–f' parallel to e'–d', where e'–d' is the vertical projection of an element

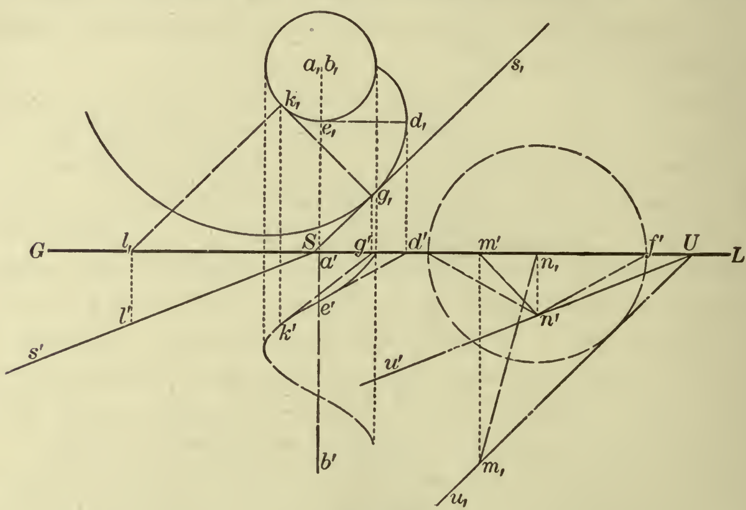

Fig. 136

of the convolute parallel to V. The elements of the right circular cone generated by the revolution of the line n'–f' about the vertical line through n' will make the same angle with H as the elements of the convolute. The circle whose center is $n_{\prime}$ is the base of the cone and n' is the vertex of the cone.

Through M–N and tangent to the cone draw the plane U (see Section 321). The horizontal trace U–$u_{\prime}$ will be tangent to the circular base. The vertical trace U–u' will pass through n'.

Tangent to the base of the convolute and parallel to $U\text{–}u_{,}$ draw $S\text{–}s_{,}$, the required horizontal trace. Through S and parallel to $U\text{–}u'$ draw $S\text{–}s'$, the required vertical trace.

The element of tangency $G\text{–}K$ may be determined by drawing the element of the surface through the point of tangency G on the base.

Since another plane may be drawn through $M\text{–}N$ and tangent to the cone, another plane may be constructed which will answer the conditions of the problem.

Since the convolute is unlimited in extent, having its base in the form of a spiral, an unlimited number of planes answering the conditions of the problem may be constructed.

Check. Through any point, as K, on the element of tangency draw a straight line $K\text{–}L$ parallel to $S\text{–}s_{,}$, and note whether it pierces V in the vertical trace already located.

CHAPTER XIV

DETERMINATION OF PLANES TANGENT TO SURFACES OF DOUBLE CURVATURE

332. General Instructions. By Section 226 straight lines which are tangent to a surface of double curvature at a point on the surface, lie in a plane tangent to the surface at this point. Therefore through the point of tangency draw two planes cutting from the surface two simple curved lines intersecting at the point of tangency. Tangent to these curved lines at the point of tangency draw two straight lines which shall determine the required plane.

When the given surface is one of revolution, it will be found convenient to use as the cutting planes the meridian plane and a plane perpendicular to the axis.

Sometimes an auxiliary surface may be passed tangent to the given surface at the point of tangency; then of course a plane tangent to the auxiliary surface at the point of tangency will also be tangent to the given surface at the point of tangency.

333. Problem 208. *To draw a plane tangent to a sphere at a point on the surface.*

Analysis 1. See Section 332.

Analysis 2. Draw a plane perpendicular to the radius of the sphere at the point of tangency.

Construction. See Fig. 137. Let C represent the sphere and let O, assumed as in Section 287, represent the point on the surface.

By Analysis 1 draw through O a vertical meridian plane T, cutting the sphere in a great circle. Revolve T about its horizontal trace into H. The center of the great circle will fall at c_H and the point of tangency will fall at o_H. Through o_H draw o_H–$a_{,}$ tangent to the circle c_H. This last line is the revolved position of a tangent to the sphere at the point O. This tangent in true position pierces H at $a_{,}$, a point in the required horizontal trace.

Through O pass another plane U parallel to H. This plane cuts the sphere in a small circle whose horizontal projection is $o_{\prime}–o_{\prime\prime}–b_{\prime}$ and whose vertical projection is $o''–b'$. Through O draw $O–D$ tangent to the small circle at O. $O–D$ is another line of the required plane, and pierces V at d', a point in the required vertical trace.

Since $O–D$ is a line of the required plane and parallel to H, its horizontal projection $o_{\prime}–d_{\prime}$ must be parallel to the required horizontal trace (see Section 42). Therefore through $a_{\prime}$ and parallel to $o_{\prime}–d_{\prime}$ draw the required horizontal trace $S–s_{\prime}$. Through S and d' draw the required vertical trace $S–s'$.

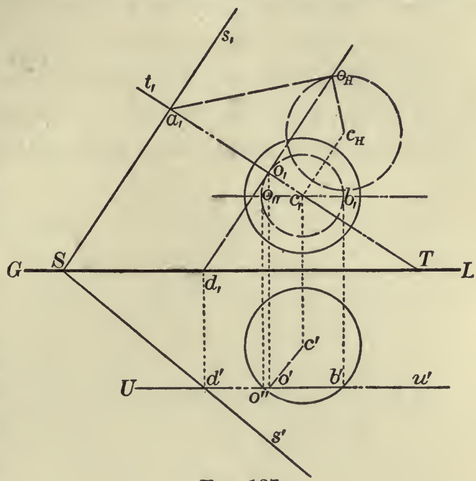

FIG. 137

Check. The traces of the required plane should be perpendicular to the projections of the radius $C–O$.

334. Problem 209. *Solve the above problem by Analysis* 2.

335. Problem 210. *Given a sphere with center in G–L; required to assume a point upon the surface and to pass a plane tangent to the surface at this point.*

336. Problem 211. *Given a sphere in the second quadrant with its center equidistant from H and V; required to assume a point upon the surface and to pass a plane tangent to the surface at this point.*

337. Problem 212. *To draw a plane tangent to an ellipsoid of revolution at a point on the surface.*

Analysis 1. See Section 332.

Analysis 2. Determine the meridian curve of the surface which contains the given point, and draw a rectilinear tangent to the curve at this point. If this tangent is revolved about the axis of the ellipsoid as an axis, it will generate a conical surface which will be tangent to the surface of the ellipsoid in the circumference of a circle containing the point of tangency (see Section 284).

A plane tangent to this cone along the element passing through the given point of tangency will also be tangent to the ellipsoid at the same point.

Construction. Let the ellipsoid be represented as in Fig. 138, where A–B represents the long axis and where O represents the point assumed on the surface.

Following Analysis 1, pass a meridian plane T, containing the point O and cutting the surface in a meridian ellipse. Revolve T about A–B as an axis until it is parallel to V. The point O will then be vertically projected at o'' and the meridian ellipse will be vertically projected in the vertical projection of the ellipsoid.

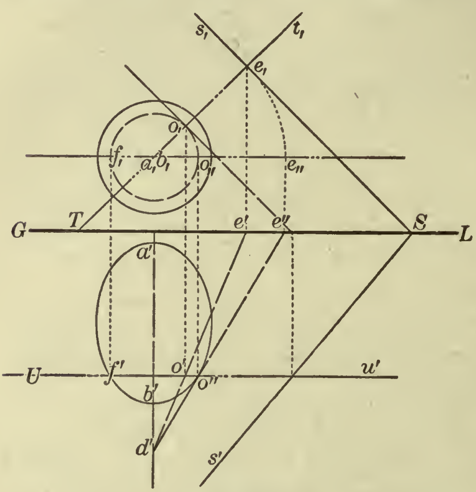

FIG. 138

Tangent to the ellipse a'–o''–b'–f' at the point o'' draw d'–o''–e''. This line is the vertical projection of the revolved position of a straight line D–E, which in true position is tangent to the surface at O, and therefore in the required plane. In revolved position D–E pierces H at e_{II}, but in true position, since D is in the axis, it will pierce H at e_I.

Through O pass another auxiliary plane U perpendicular to the axis A–B. This plane cuts the ellipsoid in a circle whose horizontal projection is f_I–o_I–o_{II} and whose vertical projection is f'–o''.

The remainder of the construction corresponds with that given in connection with the sphere (see Section 333).

Check. Construct according to Analysis 2.

338. Problem 213. *Solve the above problem by Analysis 2.*

339. Problem 214. *Given an ellipsoid of revolution with the long axis perpendicular to V; required to assume a point upon the surface and to draw a plane tangent to the surface at this point.*

340. Problem 215. *Given an ellipsoid of revolution with the long axis parallel to G–L; required to assume a point upon the surface and to draw a plane tangent to the surface at this point.*

341. Problem 216. *To draw a plane tangent to any surface of revolution.*

Analysis. The directions given in connection with the solution of the foregoing problems will be sufficient for the solution of all problems relative to tangency of planes to surfaces of revolution.

342. Problem 217. *To draw a plane containing a given straight line and tangent to a given sphere.*

Analysis 1. After the required plane is constructed, an auxiliary plane through the center of the sphere and perpendicular to the given line will cut from the sphere a great circle, from the line a point, and from the required plane a straight line passing through the point and tangent to the great circle. Therefore, through the center of the sphere and perpendicular to the given line, pass a plane cutting the sphere in a great circle and cutting the line in a point. Through this point and tangent to the circle draw a straight line. This line together with the given line will determine the required plane.

Analysis 2. If through any point of the given line a series of rectilinear tangents to the sphere be drawn, they will form the elements of a cone whose axis will contain the point on the line and the center of the sphere. The surface of this cone will be tangent to the sphere in the circumference of a small circle whose plane will be perpendicular to the axis of the cone.

A plane tangent to the cone must also be tangent to the sphere. Therefore a plane through the given line and tangent to the cone will be the required plane.

Analysis 3. If from each of two points on the given line a series of rectilinear tangents to the sphere be drawn, they will form the elements of two cones whose axes will contain the points on the line and the center of the sphere, and whose surfaces will be tangent to the sphere in the circumferences of small circles, which may be taken as the bases of the cones. The planes of these small circles will be perpendicular to the axes of their respective cones, and will intersect in a straight line which will be a chord of the sphere.

The circumferences of these two circles will, as a rule, intersect in two points on the surface of the sphere, which will be the points in which the chord of the sphere intersects the surface of the sphere, and which will be the points at which the required planes will be tangent to the sphere. For a straight line drawn through the vertex of either cone and one of these points of tangency is an element of this cone and will therefore be tangent to the sphere at this point of tangency. A straight line through the vertex of the other cone and this same point of tangency is an

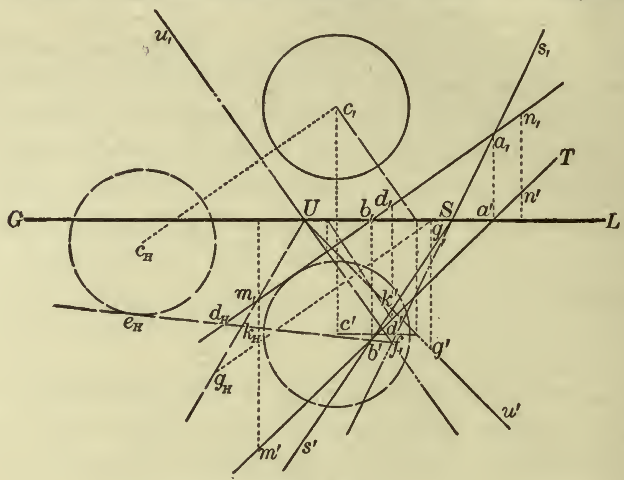

FIG. 139

element of the other cone and will therefore be tangent to the sphere at this same point of tangency.

The plane of these two tangents will contain the given line and will be tangent to the sphere at this point of tangency.

Construction 1. In Fig. 139 let C represent the center of the sphere and let M–N represent the given line.

Produce M–N to intersect H at $a_{,}$ and to intersect V at b', points in the required traces.

Following Analysis 1, pass the plane U through the point C and perpendicular to M–N (see Section 165). Find the point D in which U is intersected by M–N (see Section 151).

Revolve the plane U about its horizontal trace into H. D will fall at d_H and the center of the great circle cut from the sphere by

U will fall at c_H. Through d_H draw e_H–d_H–f_i tangent to the circle c_H at e_H. The straight line of which e_H–d_H–f_i is the revolved position passes through D and is tangent to the great circle cut from the sphere by U, and is therefore a line of the required plane. Produce this line to meet H in f_i. Through f_i and a_i draw S–s_i, the required horizontal trace. Through S and b' draw S–s', the required vertical trace.

Check. Assume any point, as G, in the vertical trace of U. After the revolution of U, G will take the position g_H, and the vertical trace U–u' will take the position U–g_H. The point k_H, the point in which U–g_H intersects e_H–f_i, is the revolved position of the point in which E–F crosses the vertical trace of U, or, in other words,

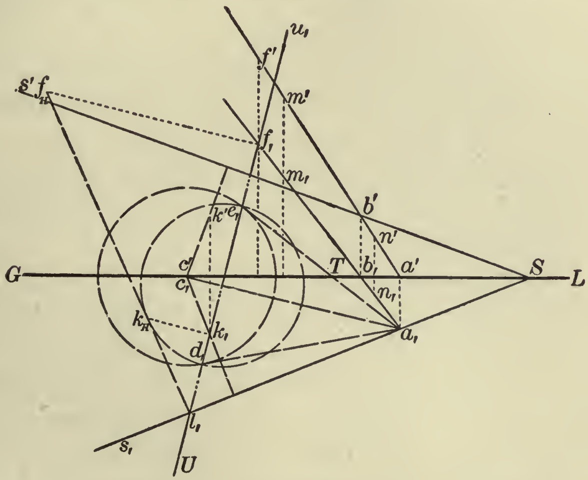

Fig. 140

k_H is the revolved position of the point in which E–F pierces V, a point necessarily in the vertical trace of the required plane.

The true position of this point K must be upon U–u' at a distance from U equal to U–k_H, and should be on S–s' already located.

Since from d_H another rectilinear tangent to the circle c_H may be drawn, another plane answering the conditions of the problem may be constructed.

Construction 2. In Fig. 140 let C represent a sphere whose center is in G–L and let M–N represent the given line.

Produce M–N to intersect H in $a_{\prime}$ and to intersect V in b', points in the required traces.

Following Analysis 2, take the point A — the point in which M–N pierces H — as the vertex of a cone whose elements are tangent to the sphere. The lines $a_{\prime}$–$d_{\prime}$ and $a_{\prime}$–$e_{\prime}$, each tangent to the horizontal projection of the sphere, will represent the horizontal projections of the extreme elements of the cone. The line $a_{\prime}$–$c_{\prime}$ represents the horizontal projection of the axis of the cone, and U–$d_{\prime}$–$e_{\prime}$–$u_{\prime}$ represents the horizontal trace of the plane of the circle of tangency, which circle may be taken as the base of the cone.

Determine the point F in which M–N intersects U. Revolve U about its horizontal trace U–$u_{\prime}$ as an axis into H. F will fall at $f_{\prime\prime}$, and the circle of tangency will fall at $d_{\prime}$–$k_{\prime\prime}$–$e_{\prime}$. Through $f_{\prime\prime}$ draw $f_{\prime\prime}$–$k_{\prime\prime}$–$l_{\prime}$ tangent to the circle $d_{\prime}$–$k_{\prime\prime}$–$e_{\prime}$ at the point $k_{\prime\prime}$.

This line is, by Section 321, the revolved position of a line of the plane containing the line M–N and tangent to the cone, and is therefore a line of the required plane.

In true position this line F–K–L pierces H at $l_{\prime}$. Through $a_{\prime}$ and $l_{\prime}$ draw the required horizontal trace S–$s_{\prime}$. Through S and b' draw the required vertical trace S–s'.

Check. The required plane is tangent to the cone along the element A–K, and since this element is tangent to the sphere at the point K, K is the point at which the required plane is tangent to the sphere. The required plane must then be perpendicular to the radius K–C. The projections of K in true position are $k_{\prime}$ and k', where the distance of k' from G–L is equal to $k_{\prime}$–$k_{\prime\prime}$. Therefore S–$s_{\prime}$ and S–s' should be perpendicular respectively to $k_{\prime}$–$c_{\prime}$ and k'–c'.

Since from the point $f_{\prime\prime}$ another tangent to the circle $d_{\prime}$–$k_{\prime\prime}$–$e_{\prime}$ may be drawn, another plane answering the conditions of the problem may be constructed.

Construction 3. See Fig. 141. C represents the center of the sphere and M–N represents the given line.

M–N pierces H at $a_{\prime}$ and pierces V at b', points in the required traces.

Following Analysis 3, take for the vertices of the cones the points D and E, which are assumed upon M–N in such a way that D is at the same distance below H as the center of the sphere,

and that E is at the same distance back of V as the center of the sphere.

The axis of the cone D is D–C and by construction is parallel to H. The axis of the cone E is E–C and by construction is parallel to V.

To represent the horizontal projection of the cone D, draw through $d_{\prime}$, and tangent to the horizontal projection of the sphere, the two lines $d_{\prime}$–$f_{\prime}$ and $d_{\prime}$–$g_{\prime}$.

To represent the vertical projection of the cone E, draw through e', and tangent to the vertical projection of the sphere, the two lines e'–k' and e'–l'.

The base of the cone D is a circle whose plane is perpendicular to H and whose horizontal projection is $f_{\prime}$–$g_{\prime}$. The base of the cone E is a circle whose plane is perpendicular to V and whose vertical projection is k'–l'.

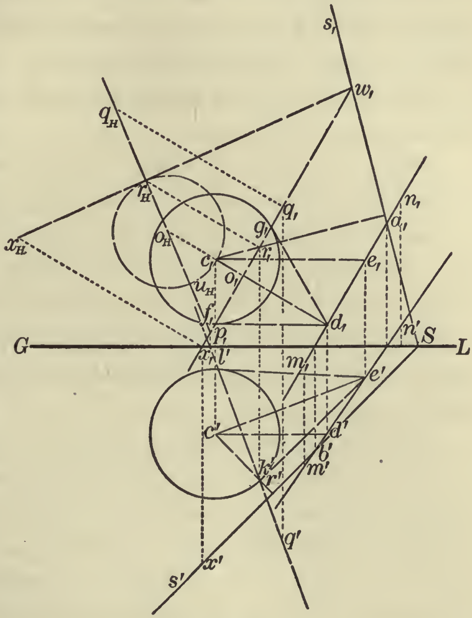

FIG. 141

The planes of these bases will intersect in a straight line whose horizontal projection will fall on $f_{\prime}$–$g_{\prime}$ and whose vertical projection will fall on k'–l'.

This line of intersection will pierce the surface of the sphere in the two points in which the circumferences of the two bases intersect, and which by analysis are the points of tangency sought.

To find these points of tangency revolve the plane of the base of the cone whose vertex is D about its horizontal trace $f_{\prime}$–$g_{\prime}$ into H. The circular base whose center is O will take the position of the circle whose center is o_H, where $o_{\prime}$–o_H is equal to the distance of c' from G–L. The intersection of the planes of the two bases

will fall at q_{II}–p_I, where p_I represents the point in which this line of intersection in true position pierces H and where Q represents any point assumed on the line.

The points r_{II} and u_{II} are then the revolved positions of the two points of tangency sought.

Through r_{II} and tangent to the circle o_{II} draw r_{II}–w_I. This is the revolved position of a line of the required plane, since it is the revolved position of a straight line tangent to the sphere at the point at which the required plane is to be tangent. The line R–W in true position pierces H at w_I and pierces V at x', where x_I–x' is equal to x_I–x_{II} measured on a straight line, x_I–x_{II}, perpendicular to w_I–x_I.

Through a_I and w_I draw the required horizontal trace S–s_I. Through b' and x' draw the required vertical trace S–s'.

Check. The two traces should cross G–L at the same point. Or, the two traces, S–s_I and S–s', should be perpendicular respectively to the two projections of the radius R–C.

Since the line Q–P intersects the surface of the sphere in another point U, another plane answering the conditions of the problem may be constructed.

343. Problem 218. *Given a sphere whose center is in G–L, and given a straight line parallel to G–L and in the third quadrant; required to draw a plane containing the line and tangent to the sphere.*

344. Problem 219. *Given a sphere whose center is in G–L, and given a straight line situated in a profile plane and in the third quadrant; required to pass a plane through the line and tangent to the sphere.*

345. Problem 220. *Given a sphere in the second quadrant equidistant from H and V, also given a straight line in the first quadrant; required to pass a plane through the line and tangent to the sphere.*

CHAPTER XV

INTERSECTION OF SURFACES BY LINES

346. General Instructions. If the given line is a straight line or a curved line of single curvature, pass an auxiliary plane through the line and determine its intersection with the given surface. The point or points in which the given line intersects the lines cut from the surface by the auxiliary plane will be the points sought.

Since an infinite number of planes may be drawn through a straight line, it will be wise, in connection with straight lines, to use those auxiliary planes which will cut from the surface the simplest lines. If the given line is a curved line of single curvature the plane of the curve must be used as the auxiliary plane.

347. On the Character of Lines in which Planes intersect Surfaces. Since prisms and pyramids have plane surface faces, the intersections of these faces by planes will always be straight lines whatever the position of the cutting plane.

Since the elements of cylindrical surfaces are parallel straight lines, the intersections of such surfaces by planes which are parallel to the elements are parallel straight lines.

Since the elements of conical surfaces all pass through the vertex, the intersections of such surfaces by planes which contain the vertex will be straight lines passing through the vertex.

The intersection of a sphere by a plane is always a circle. If the cutting plane contains the center of the sphere, the circle is a great circle.

348. Problem 221. *To find the point in which a given straight line intersects a given plane* (*see Section 151*).

349. Problem 222. *To find the points in which a given straight line intersects the surface of a square prism.*

Analysis. Pass the auxiliary plane through the line and parallel to the edges of the prism.

Construction. Let the prism be assumed with its edges perpendicular to H, as shown in Fig. 142, and let M–N represent the given line.

The auxiliary plane is taken perpendicular to H and intersects the surface in two lines, A–B and D–E, each parallel to the edges. These lines are crossed by M–N at X and Y, the two points required.

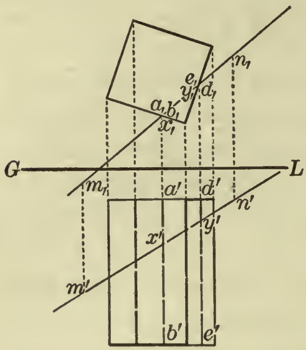

Fig. 142

350. Problem 223. *Find the points in which a straight line intersects the surface of a prism whose edges are oblique to H and V.*

351. Problem 224. *Find the points in which a straight line intersects the surface of an hexagonal prism.*

352. Problem 225. *Given a hollow square prism; required to find the points in which a given straight line intersects the outer and the inner surface of the prism.*

353. Problem 226. *To find the points in which a given straight line intersects the surface of a square pyramid.*

Analysis. Pass the auxiliary plane through the line and the vertex of the pyramid.

Construction. Let the pyramid be represented as in Fig. 143, and let M–N represent the given line. Produce M–N to meet the plane of the base of the pyramid at B. Through the vertex A and any point D of the line M–N draw a straight line and produce it to meet the plane of the base at E. The straight line through B and E is the intersection of the auxiliary plane and the plane of the base of the pyramid.

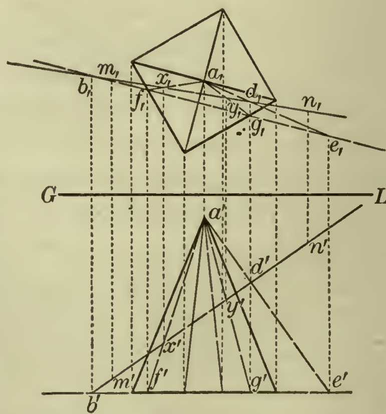

Fig. 143

This auxiliary plane intersects the pyramid in two lines, A–F and A–G. M–N crosses these lines at X and Y, the points required.

354. Problem 227. *Find the points in which a given straight line intersects the surface of an oblique hexagonal pyramid.*

355. Problem 228. *Find the points in which a given straight line intersects the surface of the frustum of a square pyramid.*

356. Problem 229. *Given a hollow square pyramid; required to find the points in which a given straight line intersects the outer and the inner surface of the pyramid.*

357. Problem 230. *To find the points in which a given straight line intersects the surface of a cylinder.*

Analysis. Pass the auxiliary plane through the line and parallel to the elements of the cylinder, or, what is the same thing, parallel to the axis of the cylinder.

Construction. Let the cylinder be represented as in Fig. 144, and let M–N represent the given line. Produce M–N to pierce H at $d_{\prime}$. Through N, any point on M–N, draw N–E parallel to the axis of the cylinder and produce it to pierce H at $e_{\prime}$. The straight line U–$d_{\prime}$–$e_{\prime}$–$u_{\prime}$, determined by the points $d_{\prime}$ and $e_{\prime}$, is the horizontal trace of the auxiliary plane.

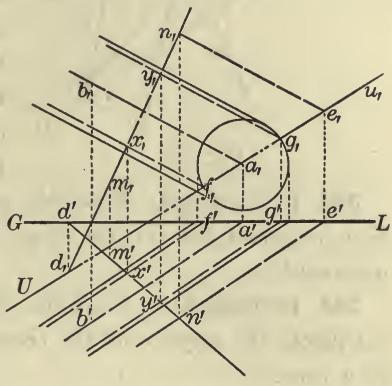

Fig. 144

This plane U cuts the cylinder in two elements, F–X and G–Y, which are intersected by M–N at X and Y, the two points required.

358. Problem 231. *Given a right circular cylinder in the third quadrant with axis parallel to G–L; required to find the points in which a given straight line intersects the surface of the cylinder.*

359. Problem 232. *Given a right circular cylinder in the third quadrant with axis perpendicular to H; required to find the points in which a given straight line intersects the surface of the cylinder.*

360. Problem 233. *Given a hollow cylinder; required to find the points in which a given straight line intersects the inner and the outer surface of the cylinder.*

361. Problem 234. *To find the points in which a given straight line intersects the surface of a cone.*

Analysis. Pass the auxiliary plane through the line and the vertex of the cone.

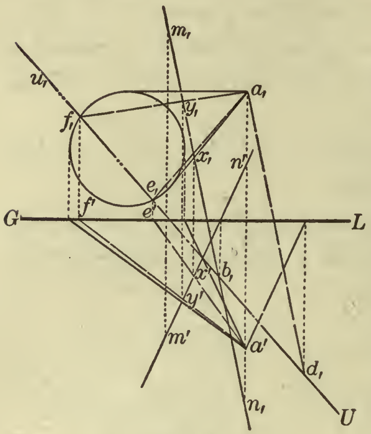

Fig. 145

Construction. Let the cone be represented as in Fig. 145, and let M–N represent the given line.

Produce M–N to pierce H at $b_{,}$. Through A, the vertex of the cone, and parallel to the given line, draw A–D piercing H at $d_{,}$. U–$d_{,}$–$b_{,}$–$u_{,}$ is the horizontal trace of the auxiliary plane. This plane intersects the surface of the cone in two elements, A–E and A–F, which are cut by M–N in X and Y, the two points required.

362. Problem 235. *Find the points in which a given straight line intersects the surface of an inverted cone situated in the third quadrant.*

363. Problem 236. *Find the points in which a given straight line intersects the surface of the frustum of a cone.*

364. Problem 237. *Given a hollow cone; required to find the points in which a given straight line intersects the inner and the outer surface of the cone.*

365. Problem 238. *To find the points in which a given straight line intersects the surface of a sphere.*

Analysis. Since the intersection of a sphere by any plane is a circle, the auxiliary plane may be passed through the line at random.

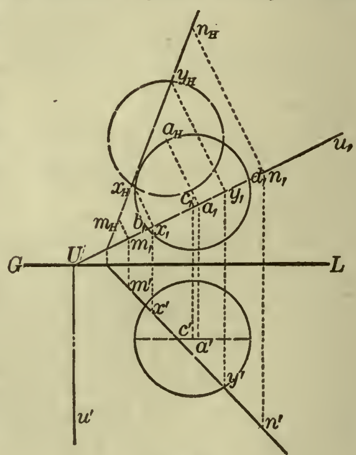

Fig. 146

Construction. Let the sphere be represented as in Fig. 146, and let M–N represent the given line.

Take as the auxiliary plane the horizontal projecting plane of the given line. This plane U cuts the sphere in a small circle whose center is A and whose diameter is $B–D$.

To determine the points in which $M–N$ intersects this circle, revolve the plane U about its horizontal trace into H. The center A of the circle cut from the sphere falls at a_H, and the line $M–N$ falls at $m_H–n_H$. The points x_H and y_H, the points in which $m_H–n_H$ intersects the circumference of the circle a_H, are the required points in revolved position. The points X and Y, whose projections are $(x_{\prime}, x')$ and $(y_{\prime}, y')$ respectively, are the required points in true position.

The problem may be as easily solved by using the vertical projecting plane of $M–N$ as the auxiliary plane.

366. Problem 239. *Given a hollow hemisphere; required to find the points in which a given straight line intersects the inner and the outer surface.*

Analysis. Take as the auxiliary plane one of the projecting planes of the given line, as in the preceding problem. Find the two semicircles in which this plane intersects the outer and inner surfaces of the hemisphere. The points in which the given line intersects these semicircles will be the required points.

CHAPTER XVI

INTERSECTION OF SURFACES BY PLANES

367. General Instructions. Pass a series of auxiliary planes intersecting both the given plane and the given surface in lines. The points in which these two sets of lines intersect will be common to both the plane and the surface, and will therefore be points in their line of intersection.

Pass the auxiliary planes in such a way as not only to cut from the given surface the simplest lines, but also in such a way as to simplify the work of construction.

If the given surface is that of a prism or of a pyramid, it will be found convenient to pass the auxiliary planes through the edges.

If the given surface is one of revolution, it will be found convenient to pass the auxiliary planes perpendicular to the axis of revolution, since such planes will cut circumferences of circles from the surface.

368. Problem 240. *To find the line in which one plane intersects another (see Section 146).*

369. Problem 241. *To find the intersection of a triangular prism by a plane, to find the true size of the intersection, and to develop the surface of the prism.*

CASE 1. *To find the intersection.*

Analysis. Pass the auxiliary planes through the edges of the prism.

Construction. In Fig. 147 let $A–B–D––E–F–G$ represent a prism with edges perpendicular to H, and let S represent the cutting plane.

The plane of the face $A–D–G–E$, which is the plane U, contains the two edges $A–E$ and $D–G$. This plane, which may be used as an auxiliary plane, cuts S in the line $K–L$, crossing $A–E$ and $D–G$ at X and Z respectively. $X–Z$ is the line of intersection between the plane S and the face $A–D–G–E$.

Pass another auxiliary plane, W, through the edge $B–F$, and locate the point Y in which the edge $B–F$ is cut by S.

S intersects the face $D–B–F–G$ in the line $Y–Z$, and intersects the face $A–B–F–E$ in the line $X–Y$. The required intersection is $X–Y–Z$.

In this case the auxiliary planes are perpendicular to H, since the edges of the prism are perpendicular to H. The method of construction is the same when the edges are oblique to H, but the auxiliary planes will then be oblique to H.

CASE 2. *To find the true size of the intersection.*

Analysis and Construction. Revolve the plane S, in Fig. 147, about $S–s_{\prime}$ as an axis into H. The three vertices X, Y, and Z will fall respectively at x_H, y_H, and z_H. The true size of the intersection is $x_H–y_H–z_H$ (see Section 87).

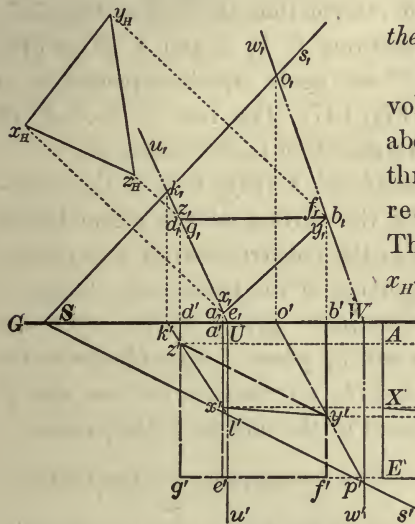

FIG. 147

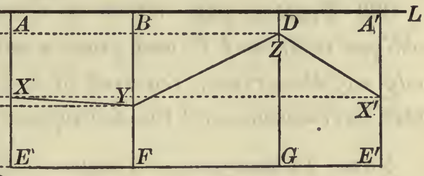

FIG. 148

The triangle $X–Y–Z$ is the exact pattern of the opening which would be necessarily made in the plane S in order to allow the prism to pass through.

CASE 3. *To develop the surface (see Section 228).*

Analysis. Take the plane of the face $A–B–F–E$ as the plane of development.

Starting with this face in the plane, roll the prism from left to right, bringing first the face $B–D–G–F$ and later the face $A–D–G–E$ into coincidence with the original plane.

Construction. Fig. 148 shows how these faces will appear after development.

Since the plane of the base of the prism is perpendicular to the edges of the prism, the straight line $A\text{–}B\text{–}D\text{–}A'$, where $A\text{–}B$, $B\text{–}D$, and $D\text{–}A'$ are made equal respectively to $a_{\prime}\text{–}b_{\prime}$, $b_{\prime}\text{–}d_{\prime}$, and $d_{\prime}\text{–}a_{\prime}$ of Fig. 147, may represent the development of the base of the surface.

The edges of the prism will, in development, take the positions $A\text{–}E$, $B\text{–}F$, $D\text{–}G$, and $A'\text{–}E'$, all perpendicular to $A\text{–}A'$ and all equal to the altitude of the prism.

The rectangle $A\text{–}A'\text{–}E'\text{–}E$ (Fig. 148) represents the developed prism.

The vertices of the triangle of intersection, $X\text{–}Y\text{–}Z$ of Fig. 147, will in development take the positions X, Y, Z, and X' (Fig. 148) where $A\text{–}X$, $B\text{–}Y$, $D\text{–}Z$, and $A'\text{–}X'$ are made equal respectively to $a'\text{–}x'$, $b'\text{–}y'$, $d'\text{–}z'$, and $a'\text{–}x'$ of Fig. 147. The line $X\text{–}Y\text{–}Z\text{–}X'$ of Fig. 148 represents the line of intersection in development.

The plane surface $E\text{–}X\text{–}Y\text{–}Z\text{–}X'\text{–}E'\text{–}E$ (Fig. 148) is the templet or pattern of that portion of the surface of the prism below the plane S, and might be used in the construction of a duplicate prism to take the place of that portion of the prism now there.

370. Problem 242. *Given a triangular prism whose edges are oblique to H and V, and given a cutting plane perpendicular to the edges of the prism; required to find the intersection, the true size of this intersection, and the development of the surface of the prism.*

NOTE. In developing this surface use the cutting plane as a basal plane.

371. Problem 243. *Given a square prism whose edges are parallel to G–L, and given a cutting plane oblique to H and V; required to find the intersection, the true size of this intersection, and the development of the surface of the prism.*

NOTE. In developing this surface use an auxiliary profile plane as a basal plane.

372. Problem 244. *Given an hexagonal prism whose edges are oblique to H and V, and given a cutting plane parallel to H; required to find the intersection, the true size of this intersection, and the development of the surface of the prism.*

NOTE. In developing this surface use an auxiliary cutting plane perpendicular to the edges of the prism, as a basal plane.

373. Problem 245. *To find the intersection of a square pyramid by a plane, to find the true size of the intersection, and to develop the surface of the pyramid.*

CASE 1. *To find the intersection.*

Analysis. Pass the auxiliary planes through the edges of the pyramid.

Construction. In Fig. 149 let $A–B–D–E–F$ represent the pyramid and let S represent the cutting plane.

Pass an auxiliary plane T through the two edges $A–B$ and $A–E$. This plane intersects S in $G–K$, which crosses the edges $A–B$ and $A–E$ at X and Z respectively.

Pass another auxiliary plane U through the two edges $A–D$ and $A–F$. U intersects S in $L–O$, which crosses the two edges $A–D$ and $A–F$ at Y and W respectively. $X–Y–Z–W$ is the required intersection.

In this particular case, on account of the character and position of the pyramid, the auxiliary planes T and U are perpendicular to H, but the method of construction is not changed when the conditions are otherwise.

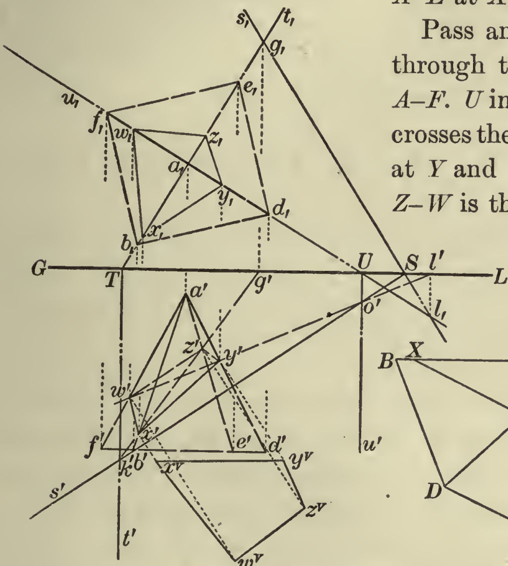

FIG. 149

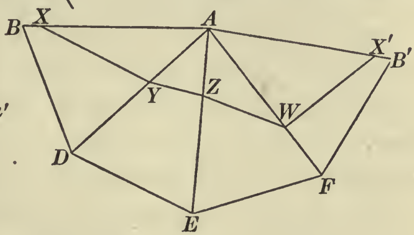

FIG. 150

CASE 2. *To find the true size of the intersection.*

Revolve the plane S in Fig. 149 about $S–s'$ into V. The points X, Y, Z, and W will fall at x^V, y^V, z^V, and w^V respectively (see Section 91), and $x^V–y^V–z^V–w^V$ will represent the true size of the intersection.

CASE 3. *To develop the surface.*

Analysis. Take the plane of the face A–B–D, Fig. 149, as the plane of development. Starting with this face in the plane, roll the pyramid from left to right, bringing first the face A–D–E, second the face A–E–F, and finally the face A–B–F, all into coincidence with the original plane.

Construction. Fig. 150 shows how these faces will appear after development.

The triangle A–B–D, Fig. 150, in which A–B, B–D, and A–D are equal respectively to A–B,* $b_{\prime}$–$d_{\prime}$, and A–D of Fig. 149, represents the face A–B–D of Fig. 149. Then since the face A–D–E, Fig. 149, is revolved about the edge A–D as an axis, it will in development take the position A–D–E, Fig. 150, where A–D, D–E, and A–E are equal respectively to A–D, $d_{\prime}$–$e_{\prime}$, and A–E of Fig. 149.

By a similar process the remainder of the figure may be constructed.

The vertices of the polygon of intersection will in development take the positions X, Y, Z, W, and X' in Fig. 150, where A–X, A–Y, A–Z, A–W, and A–X' are made equal respectively to A–X,† A–Y, A–Z, A–W, and A–X of Fig. 149.

The surface A–B–D–E–F–B', Fig. 150, represents the development of the surface of the pyramid. The broken line X–Y–Z–W–X' represents the development of the line of intersection. The surface X–B–D–E–F–B'–X'–W–Z–Y represents the development of that portion of the surface of the pyramid below the plane S.

In cases of oblique and irregular pyramids the edges of the pyramid will not be equal, neither will the sides of the base necessarily be equal, as in the case just considered.

374. Problem 246. *Given an oblique pentagonal pyramid, and given a cutting plane perpendicular to V but oblique to H; required to find the intersection, the true size of this intersection, and the development of the surface of the pyramid.*

* In Fig. 149 the distances A–B, A–D, A–E, etc., are not equal to a'–b', a'–d', a'–e', etc.

† In Fig. 149 the distances A–X, A–Y, A–Z, etc., are not equal to a'–x', a'–y', a'–z', etc.

375. Problem 247. *Given a triangular pyramid, and given a cutting plane parallel to G–L but oblique to H and V; required to find the intersection, the true size of this intersection, and the development of the surface of the pyramid.*

376. Problem 248. *Given an oblique hexagonal pyramid, and given a cutting plane parallel to H; required to find the intersection, the true size of this intersection, and the development of the surface of the pyramid.*

377. Problem 249. *To find the intersection of a right circular cylinder whose axis is perpendicular to H by a plane; to draw a rectilinear tangent to the curve of intersection; to find the true size of the intersection, and to develop the surface of the cylinder.*

CASE 1. *To find the intersection.*

Analysis 1. Pass a series of auxiliary planes parallel to the axis of the cylinder, cutting from the cylinder elements and cutting from the plane straight lines.

Analysis 2. Pass a series of auxiliary planes perpendicular to the axis, cutting from the cylinder circles and cutting from the plane straight lines.

Construction. Let the cylinder whose axis is assumed perpendicular to H be represented as in Fig. 151, and let S represent the cutting plane.

Pass an auxiliary plane T through the axis and perpendicular to S–$s_{\prime}$. T cuts the cylinder in two elements, A–B and D–E, and intersects S in a straight line F–G. F–G crosses A–B and D–E at K and E respectively, two points of the required curve of intersection.

Owing to the position which the plane T occupies with reference to S–$s_{\prime}$, the points K and E must be respectively the highest and the lowest points of the curve of intersection.

Pass another auxiliary plane U perpendicular to H and parallel to S–$s_{\prime}$. The plane U cuts the cylinder in two elements, L–M and N–O, and intersects the plane S in the straight line P–O–M. The line P–O–M crosses L–M and N–O at M and O respectively, two more points of the required curve.

By passing other auxiliary planes parallel to U we may obtain a sufficient number of points to locate the curve.

CASE 2. *To draw a rectilinear tangent to the curve of intersection.*

Analysis. A rectilinear tangent to the curve of intersection will be tangent to the surface of the cylinder at the given point, and will therefore lie in a plane tangent to the cylinder along the element passing through this point. The rectilinear tangent will also be in the plane of the curve which is the cutting plane. The required line will therefore be at the intersection of these two planes.

Construction. Let it be required to draw a rectilinear tangent to the curve of intersection at the point O, Fig. 151.

The plane tangent to the cylinder along the element through O is the plane W. The intersection of this plane with S is Q–O–R, the required tangent.

A rectilinear tangent like this can be drawn at any point on the curve before the curve itself has been drawn.

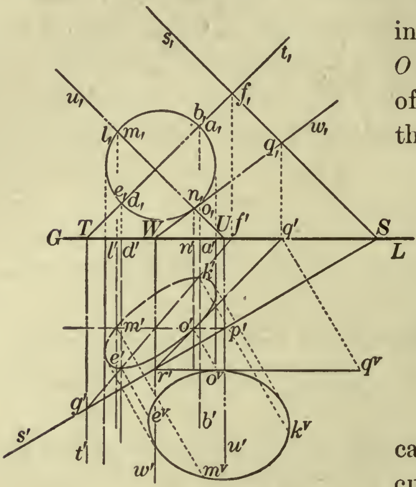

FIG. 151

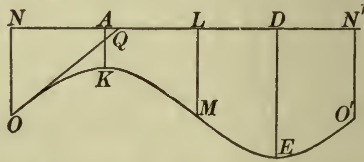

FIG. 152

Since a rectilinear tangent to a curve shows the direction of the curve at the point of tangency, such tangents, when determined in sufficient number, are helpful in drawing the curve through the points already located.

CASE 3. *To find the true size of the intersection.*

Analysis and Construction. See Fig. 151. Revolve the plane S, which is the plane of the curve of intersection, about S–s' into V. Any point, as M, will fall at m^V, and other points may be found in the same way. The curve determined by the revolved positions of these points is the required curve of intersection.

The tangent Q–O will, when revolved, take the position q^V–o^V–r' and be tangent to the revolved position of the curve of intersection.

CASE 4. *To develop the surface.*

Analysis and Construction. Take the plane W, Fig. 151, as the plane of development. Starting with the element of tangency $N–O$ in this plane, roll the cylinder along the plane away from V, bringing the successive elements into contact with W.

Since the plane of the base of the cylinder is perpendicular to the elements, the curve of the base will roll out into a straight line perpendicular to the elements.

The character of the development is shown in Fig. 152, where $N–O$ represents the element of tangency, and where $N–N'$ drawn perpendicular to $N–O$ and made equal to the rectification of the curve of the circular base of the cylinder represents the development of the base.

The elements through A, L, D, etc., in Fig. 151, will take the positions $A–K$, $L–M$, $D–E$, etc., in Fig. 152, where $N–A$, $A–L$, $L–D$, etc., are made equal respectively to the rectified arcs $n_{\prime}–a_{\prime}$, $a_{\prime}–l_{\prime}$, $l_{\prime}–d_{\prime}$, etc., of Fig. 151.

The points O, K, M, E, O, Fig. 151, which are situated upon the elements just named, will in development take the positions O, K, M, E, O' in Fig. 152, where $N–O$, $A–K$, $L–M$, etc., are made equal respectively to the distances of the same name in Fig. 151.

The development of the curve of intersection is represented by $O–K–M–E–O'$.

The tangent $O–Q$ of Fig. 151 will in development take the position $O–Q$ in Fig. 152, where $N–Q$ is made equal to $n_{\prime}–q_{\prime}$ of Fig. 151.

378. Problem 250. *Given a right circular cylinder whose axis is perpendicular to H, and given a cutting plane; required to find the intersection by passing the auxiliary planes parallel to H.*

379. Problem 251. *Given a right circular cylinder whose axis is perpendicular to H, and given a cutting plane which is parallel to $G–L$ but oblique to H and V; required to find the intersection, to draw a rectilinear tangent to the curve of intersection, to find the true size of the intersection, and to develop the surface of the cylinder.*

380. Problem 252. *Solve the above problem when the cutting plane is perpendicular to V but oblique to H.*

381. Problem 253. *To find the intersection of an oblique cylinder by any plane, to draw a rectilinear tangent to the curve of intersection, to show the true size of the intersection, and to develop the surface of the cylinder.*

CASE 1. *To find the intersection.*

Analysis. Pass the auxiliary cutting planes parallel to the axis of the cylinder and perpendicular to H.

Construction. Let the cylinder be represented as in Fig. 153, and let S, in this case assumed perpendicular to the elements of the cylinder, represent the given cutting plane.

Pass an auxiliary plane T through the axis and perpendicular to H. This plane cuts the cylinder in two elements, $A–B$ and $D–E$, and intersects the plane S in $F–G$. $F–G$ crosses $A–B$ and $D–E$ at B and E respectively, two points in the required curve of intersection.

Owing to the position of the plane T with reference to $S–s_{\prime}$, the points B and E will be respectively the lowest and the highest points of the curve of intersection.

Pass another auxiliary plane U parallel to T. This plane cuts the cylinder in two elements, $K–L$ and $M–N$, and intersects the plane S in $O–P$, where $O–P$ is necessarily parallel to $F–G$. $O–P$ crosses $K–L$ and $M–N$ at L and N respectively, two more points in the required curve.

By passing other auxiliary planes parallel to T we may obtain a sufficient number of points to locate the required curve.

This method of construction applies equally well when the given plane S is oblique to the elements of the cylinder.

CASE 2. *To draw a rectilinear tangent to the curve of intersection.*

Analysis. See Problem 249, Case 2.

Construction. See Fig. 153. Let it be required to draw a rectilinear tangent to the curve of intersection at the point L. Draw the plane W tangent to the cylinder along the element $K–L$ (see Section 302). The intersection of the planes W and S is $Q–L–R$, the required tangent.

CASE 3. *To find the true size of the intersection.*

Analysis and Construction. See Fig. 153. Revolve the plane S, which is the plane of the curve, about $S–s_{\prime}$ into H. Any point, as B,

will fall at b_H, and other points may be found in the same way. The curve traced through the points thus found is the curve required.

The tangent R–L will, when revolved, take the position $r_{\prime}$–l_H and be tangent to the revolved position of the curve.

Case 4. *To develop the surface.*

Analysis. When the surface of a cylinder is rolled out upon a plane tangent to the surface, any section of the surface made by a plane perpendicular to the elements will roll out into a straight line perpendicular to the original element of tangency. The work of development can therefore be much simplified by making a right section of the surface and using this line in development as a line of reference.

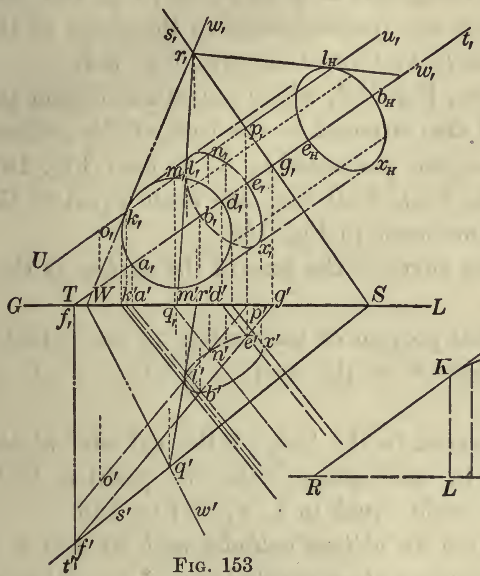

Fig. 153

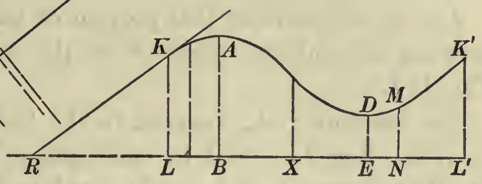

Fig. 154

Construction. Take the plane W, Fig. 153, as the plane of development, and take the point of observation to the left of the plane. Starting with the element of tangency K–L, in this plane, roll the cylinder along this plane toward V, bringing the successive elements of the surface into contact with W.

Since the plane S is perpendicular to the elements, the curve L–B–X–E–N–L will roll out into a straight line perpendicular to the elements.

The character of the development is shown in Fig. 154, where K–L represents the element of tangency and where L–L',

perpendicular to K–L and equal in length to the rectification of the true size of the right section of the surface made by the plane S, represents the development of this section.

The points L, B, X, E, etc., of Fig. 153 will, in Fig. 154, take the positions L, B, X, E, etc., where the distances L–B, L–X, L–E, etc., are made equal to the rectified arcs l_H–b_H, l_H–b_H–x_H, l_H–b_H–x_H–e_H, etc., of Fig. 153.

The elements through the points L, B, E, N, etc., Fig. 153, will in development, since they are perpendicular to the plane of the section, take the positions L–K, B–A, E–D, etc., Fig. 154.

The points K, A, D, etc., Fig. 153, which are situated upon the elements just named and also situated in the base of the surface, will in development take the positions K, A, D, etc., Fig. 154, where the distances L–K, B–A, E–D, etc., are made equal to the actual distances of the same name in Fig. 153.

The development of the curve of the base of the surface is then K–A–D–M–K', Fig. 154.

The development of that portion of the surface of the cylinder between H and the plane S is the surface K–A–D–K'–L'–L, Fig. 154.

The tangent $r_{,}$–$k_{,}$, tangent to the base of the cylinder at the point $k_{,}$, Fig. 153, will in development take the position R–K, Fig. 154, where L–R is made equal to l_H–$r_{,}$ of Fig. 153.

382. Problem 254. *Given an oblique cylinder with its axis in a profile plane, and given a cutting plane parallel to G–L but oblique to H and V; required to find the intersection, to draw a rectilinear tangent to the curve of intersection, to find the true size of the intersection, and to develop the surface of the cylinder.*

383. Problem 255. *Given a right circular cylinder with its axis parallel to G–L, and given a cutting plane oblique to H and V; required to find the intersection, to find the true size of the intersection, and to develop the surface of the cylinder.*

384. Problem 256. *Given an oblique cylinder and a cutting plane oblique to the elements of the cylinder; required to find the intersection, to draw a rectilinear tangent to the curve of intersection, to find the true size of the intersection, and to develop the surface of the cylinder.*

385. Problem 257. *To find the intersection of a right circular cone by an oblique plane, to draw a rectilinear tangent to the curve of intersection, to find the true size of the intersection, and to develop the surface of the cone.*

CASE 1. *To find the intersection.*

Analysis 1. Pass the auxiliary planes through the axis of the cone.

Analysis 2. Pass the auxiliary planes perpendicular to the axis.

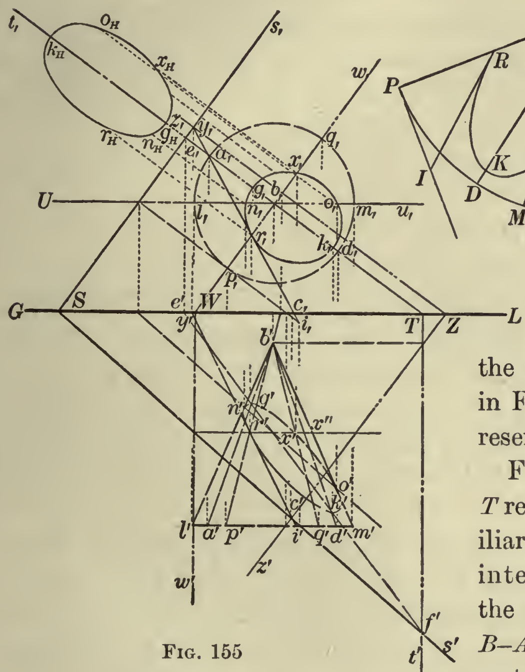

FIG. 155

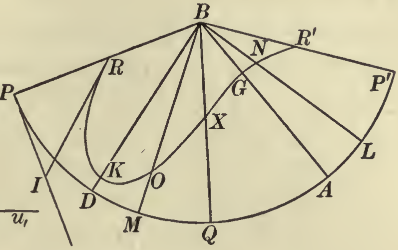

FIG. 156

Construction. Let the cone be represented as in Fig. 155, and let S represent the cutting plane.

Following Analysis 1, let T represent one of the auxiliary planes. This plane intersects the surface of the cone in two elements, B–A and B–D, and intersects S in the line E–F. E–F crosses B–A and B–D at G and K respectively, two points in the required curve of intersection.

Another auxiliary plane U parallel to V will intersect the surface in the two extreme elements B–L and B–M and will locate the two points N and O.

Another auxiliary plane W, parallel to S–$s_{,}$, will intersect the surface of the cone in the two elements B–P and B–Q and will locate the two points R and X.

CASE 2. *To draw a rectilinear tangent to the curve of intersection.*

Analysis. See Problem 249, Case 2.

Construction. See Fig. 155. Let it be required to draw a rectilinear tangent to the curve at the point R.

Draw the plane Z tangent to the cone along the element $B–R–P$. The intersection of the planes Z and S is the required tangent.

This intersection pierces H at $y_{,}$, pierces V at c', pierces the plane of the base at I, and must also pass through R. Therefore $Y–R–C–I$ is the tangent sought.

CASE 3. *To find the true size of the intersection.*

Analysis and Construction. See Fig. 155. Revolve the plane S about $S–s_{,}$ into H. After revolution any point, as N, will fall at n_H, and other points may be found in the same way.*

The true size of the curve of intersection is represented by $n_H–g_H–x_H–o_H–k_H–r_H$.

CASE 4. *To develop the surface.*

Analysis and Construction. Take the plane Z, Fig. 155, as the plane of development. Starting with the element of tangency, $B–P$, in this plane, roll the cone along this plane toward V, bringing the successive elements into contact with the plane Z.

Since the cone is one of revolution and the plane of the base is perpendicular to the axis, the curve of the circular base will roll out into the arc of a circle whose radius is equal to the slant height of the cone.

The character of the development is shown in Fig. 156, where B represents the vertex of the cone. The arc $P–D–M–Q–A–L–P'$, with center at B, with radius equal to the slant height of the cone, and equal in length to the rectification of the curve of the circular base of the cone, represents the development of the base.

To represent the elements in development, make $P–D$, $P–M$, $P–Q$, etc., Fig. 156, such that the rectification of their arcs shall be equal to the rectification of the corresponding arcs of the same name upon the base of the original cone, and connect these points with B.

To develop the curve of intersection, lay off from B on the elements now in development, Fig. 156, the distances $B–R$, $B–K$, $B–O$,

* In the revolution of these points, distances from the axis $S–s_{,}$ have all been diminished by a constant quantity.

etc., equal respectively to the distances of the points R, K, O, X, etc., on the surface of the cone, from the vertex, and trace the curve R–K–O–X–G–N–R' through these points.

In Fig. 155 the true distance from B to N is expressed by the distance b'–n', since the element B–L is parallel to V. The distance B–O is expressed by b'–o' for the same reason.

The distance from B to any other point on the surface may be found by revolving the cone about its axis until the element on which the point stands is parallel to V. The vertical projection of the required distance in this position will be equal to the distance itself.

For example, suppose it is required to find the distance B–X on the element B–Q. After revolution the element B–Q will take the position B–M, vertically projected at b'–m'. The point x' will take the position x'' and b'–x'' will be the measure of the distance B–X.

The tangent R–I of Fig. 155 will in development take the position R–I, Fig. 156, where P–I is drawn tangent to P–D–M–A–P' at P, and where P–I is made equal to $p_{\prime}$–$i_{\prime}$, Fig. 155.

386. Problem 258. *Given a right circular cone with axis perpendicular to H, and given an intersecting plane oblique to H and V; required to find the intersection by passing auxiliary planes parallel to H.*

387. Problem 259. *Given a right circular cone with axis perpendicular to H, and given an intersecting plane parallel to H; required to determine the character of the intersection.*

388. Problem 260. *Given a right circular cone with axis perpendicular to H, and given an intersecting plane passing through the axis; required to determine the character of the intersection.*

389. Problem 261. *Given a right circular cone with axis perpendicular to H, and given an intersecting plane making a smaller angle with H than the elements of the cone; required to determine the character of the intersection.*

390. Problem 262. *Solve Problem 261 when the intersecting plane makes the same angle with H as the elements of the cone.*

391. Problem 263. *Solve Problem 261 when the intersecting plane makes a greater angle with H than the elements of the cone.*

392. Conic Sections. By an examination of the results obtained in Problems 259 and 260 it will be observed that when the surface

of a right circular cone is cut by a plane perpendicular to the axis, the curve of intersection is the circumference of a circle; and when cut by a plane containing the axis, the intersections are elements.

By examination of the results obtained in Problems 261–263, and by reference to treatises on solid and analytic geometry, it may be shown that when a right circular cone with axis perpendicular to H is cut by a plane making a smaller angle with H than the elements of the cone, the curve of intersection is an *ellipse;* when cut by a plane making the same angle with H as the elements of the cone, the curve of intersection is a *parabola;* and when cut by a plane making a greater angle with H than the elements of the cone, the curve of intersection is a *hyperbola.*

393. Problem 264. *To find the intersection of any cone by a plane, to draw a rectilinear tangent to the curve of intersection, and to find the true size of the intersection.*

CASE 1. *To find the intersection.*

Analysis and Construction. Let the cone be represented as in Fig. 157, and let S represent the intersecting plane.

Pass the auxiliary planes through the vertex B and perpendicular to H. Then all the auxiliary planes will intersect in a common straight line through the vertex and perpendicular to H. This straight line will intersect S at A (see Section 151), which is therefore a point common to all the lines cut from S by the auxiliary planes.

Let T represent one of the auxiliary planes. This plane intersects the surface of the cone in two elements, B–D and B–E, and intersects the plane S in the line F–A, passing through the point A previously located. F–A crosses B–D and B–E at K and L respectively, two points in the required curve of intersection.

Another auxiliary plane U will intersect the surface of the cone in the elements B–M and B–N, and will intersect the plane S in the line I–A. I–A crosses B–M and B–N at O and P respectively, two more points in the required curve.

CASE 2. *To draw a rectilinear tangent to the curve of intersection.*

Analysis. See Problem 249, Case 2.

Construction. See Fig. 157. Let it be required to draw a rectilinear tangent to the curve at the point L. Draw the horizontal

trace of the plane W tangent to the cone along the element $B–L–E$ (see Section 317). The intersection of the planes W and S is the required tangent. This intersection pierces H at $q_{\prime}$ and must also pass through L. Therefore $Q–L$ is the required tangent.

CASE 3. *To find the true size of the intersection.*

Analysis and Construction. See Fig. 157. Revolve the plane S about $S–s_{\prime}$ into H.

After revolution any point, as O, will fall at o_H, and other points may be found in the same way.

Or revolve A, the point common to all the lines cut from S by the auxiliary planes, about $S–s_{\prime}$ into H. Then all of the lines cut

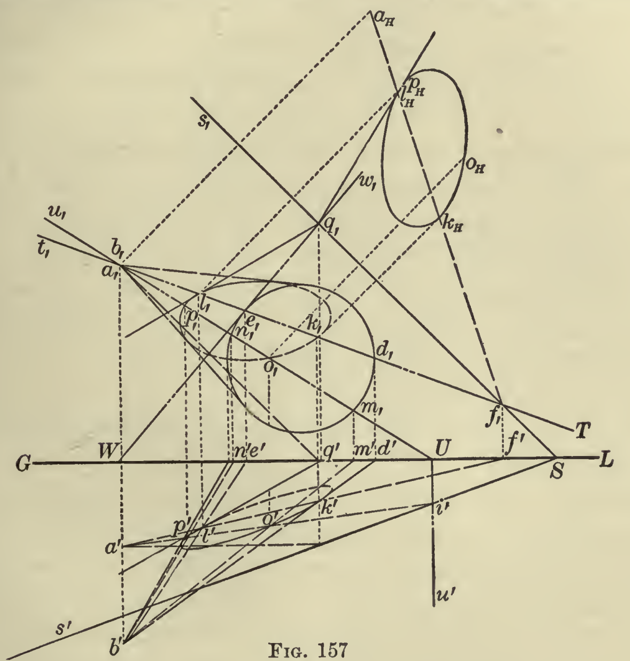

FIG. 157

from S by the auxiliary planes will in revolved position pass through the revolved position of this point.

For example, $A–F$ will in revolved position fall at $a_H–f_{\prime}$. The points K and L upon this line $A–F$ will move in planes perpendicular to $S–s_{\prime}$ and will therefore fall at k_H and l_H respectively, upon lines through $k_{\prime}$ and $l_{\prime}$ perpendicular to $S–s_{\prime}$.

The true size of the curve of intersection is represented by l_{II}–p_{II}–o_{II}–k_{II}.

The tangent Q–L will when revolved take the position q_{I}–l_{II}, the point q_{I} remaining stationary.

The development of an oblique cone requires knowledge of a problem not yet explained, and will be taken up later (see Section 421).

394. Problem 265. *Given an oblique cone with base on V, and given a cutting plane oblique to G–L; required to find the intersection, to draw a rectilinear tangent to the curve of intersection, and to find the true size of the intersection.*

395. Problem 266. *Given an oblique cone with axis in a profile plane, and given an intersecting plane parallel to G–L but oblique to H and V; required to find the intersection, to draw a rectilinear tangent to the curve of intersection, and to find the true size of the intersection.*

396. Problem 267. *Given a right circular cone with axis parallel to G–L, and given a cutting plane oblique to G–L; required to find the intersection, to draw a rectilinear tangent to the curve of intersection, and to find the true size of the intersection.*

397. Problem 268. *To find the intersection of any surface of revolution by a plane, to draw a rectilinear tangent to the curve of intersection, and to find the true size of the intersection.*

CASE 1. *To find the intersection.*

Analysis. Pass the auxiliary planes perpendicular to the axis (see Section 284).

Construction. Let the surface of revolution be represented as in Fig. 158, where A–B is the axis and where S is the cutting plane.

Let T represent one of the auxiliary planes perpendicular to A–B. This plane cuts the given surface in the circumference of a circle whose vertical projection is d'–e' and whose horizontal projection is d_{I}–g_{I}–e_{I}–k_{I}. This same plane intersects S in the straight line G–K–F parallel to S–s_{I}. G–K–F crosses the circumference D–G–E–K at G and K, two points of the required curve of intersection.

Pass a meridian plane U perpendicular to S–s_{I}, cutting from the surface a meridian curve and cutting S in a straight line M–L which intersects the axis at M.

To find the points in which M–L intersects the meridian curve, revolve the plane U about the axis A–B until it is parallel to V. The vertical projection of the meridian curve will then be identical with the vertical projection of the surface, and the vertical projection of M–L will fall at m'–l'', M remaining stationary. The points n'' and o'', the points in which the line m'–l'' crosses the vertical projection of the meridian curve, are the vertical projections of

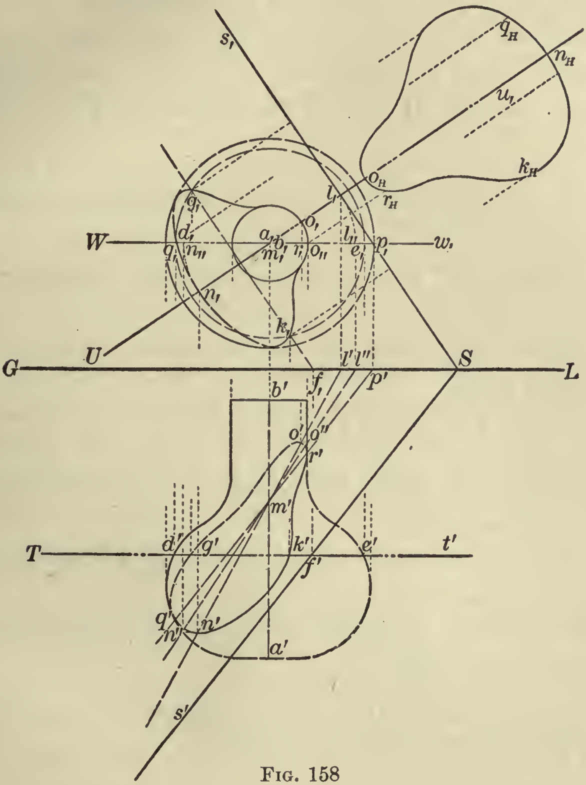

Fig. 158

the revolved positions of the lowest and the highest points of the required curve of intersection. The projections of N in true position will fall at $n_{,}$ and n', and the projections of O in true position will fall at $o_{,}$ and o'.

A meridian plane W, parallel to V, will cut from the surface a meridian curve whose vertical projection is the vertical projection

of the surface, and will cut from S a straight line whose vertical projection is $p'-m'$, parallel to $S-s'$. By this plane W two more points, Q and R, are located.

CASE 2. *To draw a rectilinear tangent to the curve.*

By analyses previously given, a rectilinear tangent to the curve, cut from any surface by a plane, may be drawn whenever a plane can be drawn tangent to the surface at the point of tangency.

CASE 3. *To find the true size of the intersection.*

Analysis and Construction. See Fig. 158. Revolve the plane S, the plane of the curve, about $S-s_{\prime}$ into H. Any point of the curve, as O, will fall at o_H, and other points may be found in the same way.*

NOTE. In the following six problems apply the principles suggested by the analysis in Problem 268, Case 1.

398. Problem 269. *Find the intersection of a cylinder of revolution by an oblique plane.*

399. Problem 270. *Find the intersection of a cone of revolution by an oblique plane.*

400. Problem 271. *Find the intersection of a sphere by a plane oblique to G–L.*

401. Problem 272. *Find the intersection of an ellipsoid of revolution by an oblique plane.*

402. Problem 273. *Find the intersection of a paraboloid of revolution by an oblique plane.*

403. Problem 274. *Find the intersection of a hyperboloid of revolution by an oblique plane.*

* In this revolution the radii have all been shortened by a constant quantity in order to reduce the space occupied by the drawing.

CHAPTER XVII

INTERSECTION OF SURFACES BY SURFACES

404. General Instructions. Pass a series of auxiliary planes so as to cut lines from the two surfaces. The points in which the lines of one surface intersect the lines of the other are necessarily in both surfaces and therefore in their line of intersection.

Pass the auxiliary planes in such a way as not only to cut from the surfaces the simplest lines but also in such a way as to make the work of construction as simple as possible.

If the two surfaces are of revolution, with intersecting axes, it will often be found convenient to pass a series of auxiliary cutting spheres with centers at the intersection of the axes. A sphere of this character will cut from the two surfaces circumferences of circles which, lying on the surface of the sphere, will as a rule intersect. These points of intersection must lie on the line in which the given surfaces intersect, since they are common to both surfaces.

405. Problem 275. *To find the intersection of a cylinder and a cone, to draw a rectilinear tangent to the curve of intersection, and to develop the surfaces.*

CASE 1. *To find the intersection.*

Analysis. Pass the auxiliary planes through the vertex of the cone and parallel to the axis of the cylinder.

Construction. Let the cylinder be represented with its base on H, and let the cone be represented with its base on V, as shown in Fig. 159. The axis of the cylinder is represented by A–B and the axis of the cone is represented by C–D.

A straight line through the vertex of the cone parallel to the axis of the cylinder will be common to all the auxiliary planes.

Through D draw D–E parallel to A–B and produce it to meet H at $e_{\prime}$ and to meet V at f'. The horizontal traces of all the auxiliary planes must pass through $e_{\prime}$, and the vertical traces of all the auxiliary planes must pass through f'.

Through $e_{,}$ draw any horizontal trace, as S–$s_{,}$, cutting the base of the cylinder. Through S and f' draw the vertical trace S–s'. S is one of the auxiliary planes and cuts the cylinder in two elements, G–K–L and M–N–O; and cuts the cone in two elements, P–D and Q–D.

The element G–K–L of the cylinder crosses the two elements of the cone at K and L, two points of the required curve of intersection. The element M–N–O of the cylinder crosses the two elements of the cone at N and O, two more points of the curve.

In this way we may pass any number of auxiliary planes and obtain any number of points on the required curve of intersection.

If it is desired to obtain a point of the curve upon any particular element of the cylinder or of the cone, we have but to draw the traces of the auxiliary plane through the points in which this element pierces the corresponding planes of projection.

It will be noticed that since the horizontal and vertical projections of the curve of intersection are found independently, the accuracy of the work may be tested by noting whether the horizontal and vertical projections of the several points of the curve lie in straight lines perpendicular to G–L.

When projecting on H, that portion of the curve which lies upon the upper surface of the cylinder and also upon the upper surface of the cone, and between extreme elements, will be visible and should be so represented.

When projecting on V, that portion of the curve which lies upon the front surface of the cylinder and also upon the front surface of the cone, and between extreme elements, will be visible and should be so represented.

Case 2. *To draw a rectilinear tangent to the curve of intersection at a given point.*

Analysis. Since the curve of intersection lies upon both surfaces, the required tangent must lie both in a plane tangent to the cone at the given point, and in a plane tangent to the cylinder at the same point. Therefore draw two planes, one tangent to the cylinder and the other tangent to the cone at the given point. The intersection of these two planes is the required tangent.

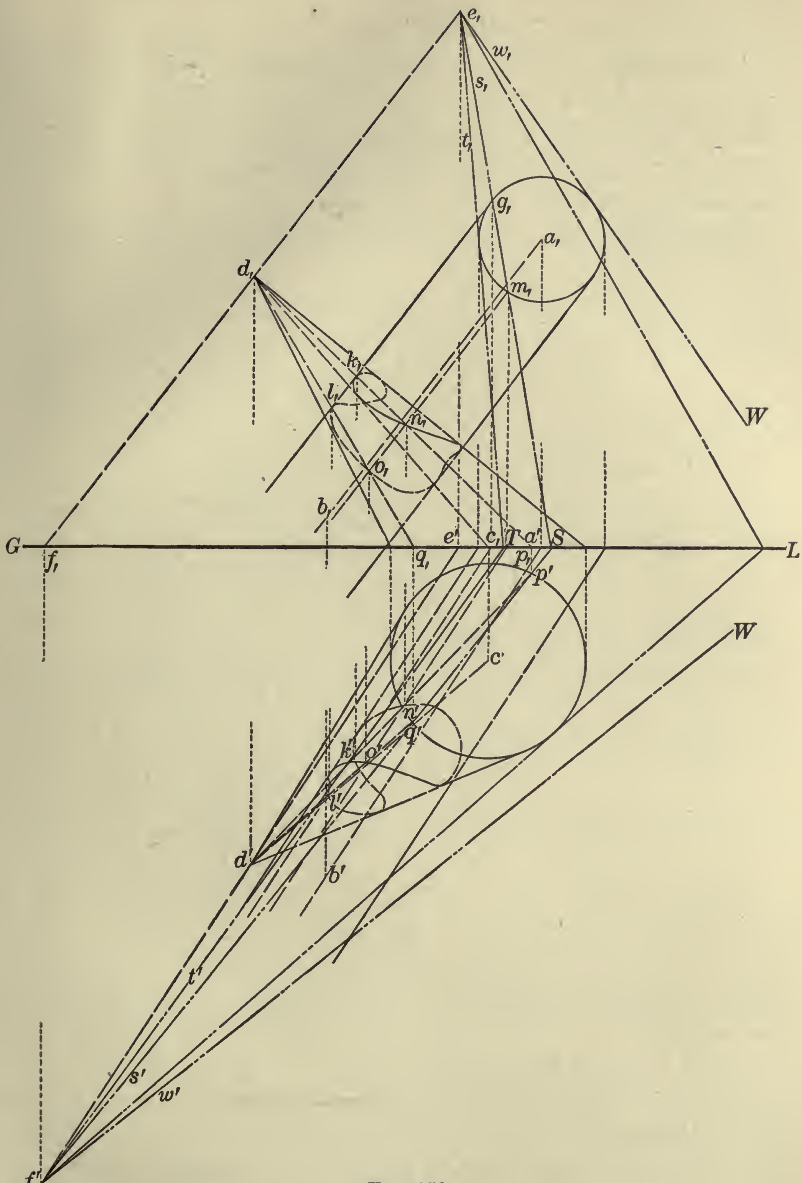

Fig. 159

CASE 3. *To develop the surfaces.*

The surface of the cylinder and the curve which lies upon it may be developed by methods previously given (see Section 377). The surface of the cone and the same curve which also lies upon this surface may be developed by the method given in Section 421, yet to be explained.

406. Problem 276. *To find the intersection of two cylinders, to*

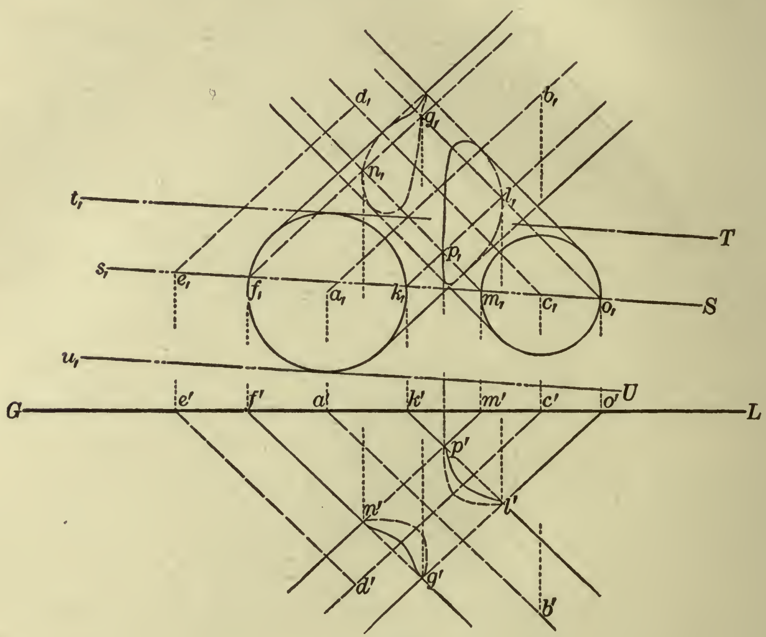

FIG. 160

draw a rectilinear tangent to the curve of intersection, and to develop the surfaces.

CASE 1. *To find the intersection.*

Analysis. Pass the auxiliary planes parallel to the axes of the two cylinders.

Construction. Let the two cylinders be represented with their two bases on H, as shown in Fig. 160. The axis of one cylinder is represented by $A–B$, and the axis of the other cylinder is represented by $C–D$.

A plane through one of the axes and parallel to the other will itself be one of the auxiliary planes, and will be parallel to all the other auxiliary planes.

Through any point on the axis $C–D$, as D, draw $D–E$ parallel to the other axis $A–B$. $C–D$ pierces H at $c_{/}$ and $D–E$ pierces H at $e_{/}$. $S–c_{/}–e_{/}–s_{/}$ is the horizontal trace of an auxiliary plane, and one to which all the other auxiliary planes must be parallel.

The plane S cuts the cylinder whose axis is $A–B$ in two elements, $F–G$ and $K–L$, and cuts the other cylinder in two elements, $M–N$ and $O–G$. These elements cross in the points N, G, L, and P, four points in the required curve of intersection.

Other auxiliary planes parallel to S may be drawn, and a sufficient number of points may be located to determine the curve of intersection.

If it is desired to determine a point upon any particular element of either cylinder, we have but to pass the auxiliary plane so as to cut from the cylinder this element.

When projecting on H, that portion of the curve of intersection which lies on the upper surfaces of both cylinders and between extreme elements will be visible.

When projecting on V, that portion of the curve of intersection which lies on the front surfaces of both cylinders and between extreme elements will be visible.

CASE 2. *To draw a rectilinear tangent to the curve of intersection at a given point.*

Analysis. See Problem 275, Case 2. The required tangent will be the intersection of two planes, — one tangent to one cylinder at the point in question, and the other tangent to the other cylinder at the same point. Therefore draw two planes under these conditions and find their line of intersection.

CASE 3. *To develop the surfaces.*

The surfaces of the cylinders and the curves of intersection which are common to both surfaces may be developed by rules previously given (see Problem 253, Case 4).

407. Problem 277. *To find the intersection of two cones, to draw a rectilinear tangent to the curve of intersection, and to develop the surfaces.*

CASE 1. *To find the intersection.*

Analysis. Draw the auxiliary planes through the two vertices.

Construction. See Fig. 161. Let A–B represent the axis of the first cone, which is right circular with base parallel to H; and let C–D represent the axis of the second cone, whose base is a circle with its plane perpendicular to V. The vertical trace of the plane of the base of the second cone is S–s', and the intersection of this plane with the plane of the base of the first cone is the line U–G–W.

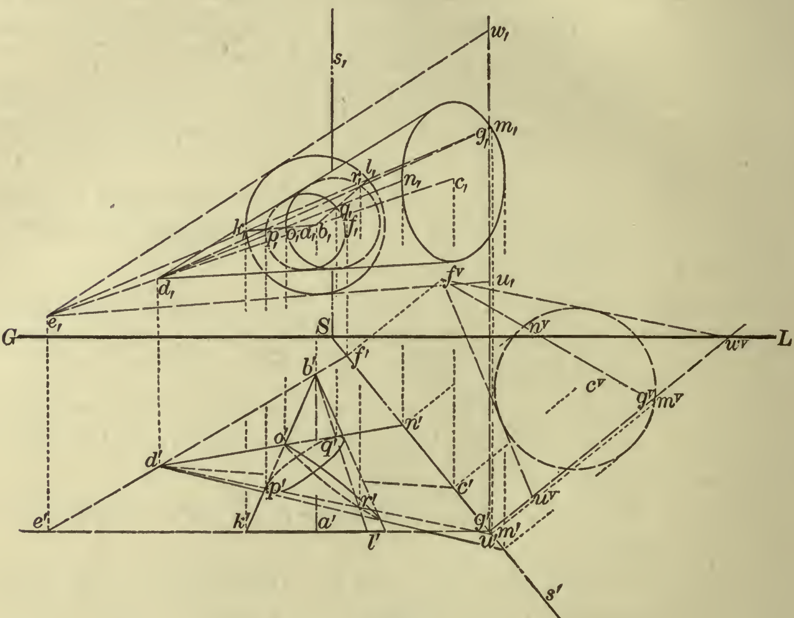

FIG. 161

A straight line through the two vertices will be common to all the auxiliary planes. Therefore through the two vertices draw B–D and produce it to meet the plane of the base of the first cone at E, a point common to all the lines cut from this plane by the auxiliary planes. This line B–D pierces S, the plane of the base of the second cone, in F, a point common to all the lines cut from S by the auxiliary planes.

Through $e_{\prime}$ draw $e_{\prime}$–$g_{\prime}$ to represent the horizontal projection of the intersection of one of the auxiliary planes with the plane of

the base of the first cone. This auxiliary plane cuts from the first cone the two elements $B-K$ and $B-L$.

The same auxiliary plane intersects the plane S in the line $F-G$, which must cross the base of the second cone in points of the elements cut from the second cone by this auxiliary plane.

To find these elements, revolve S about $S-s'$ into V. The center of the circular base of the second cone will fall at c^V, the point F will fall at f^V, and the point G will fall at g^V.

The line $f^V-g^V-m^V$ represents the revolved position of $F-G$, and the points n^V and m^V represent the revolved positions of the two points in which $F-G$ crosses the base of the second cone.

After the counter revolution M and N will take the positions $(m_{,}, m')$ and $(n_{,}, n')$ respectively, where $m_{,}$ and $n_{,}$ are at the same distances from $G-L$ as M and N respectively are from the axis of revolution $S-s'$.

The two elements cut from the second cone by this auxiliary plane are then $D-M$ and $D-N$. These elements cross the two elements cut from the first cone by this same plane at O, P, Q, and R, four points of the required curve of intersection.

Other auxiliary planes may be drawn in the same way, and a sufficient number of points may be determined to locate the required curve of intersection.

That which has been said in previous sections regarding the determination of visible and invisible portions of the curve of intersection, the construction of rectilinear tangents to this curve, and the development of the surfaces, may be said of these surfaces also.

408. To determine in advance the Nature of the Curves in which Cylinders and Cones intersect. The nature of the curve of intersection will depend upon the relative size and position of the intersecting surfaces. The surfaces may intersect so that a portion of one will remain entirely outside of the other, giving one continuous curve of intersection, as in Fig. 159; or they may intersect so as to have one distinct curve of ingress and another distinct curve of egress, as in Fig. 160; or the two surfaces may lie between two planes to which the surfaces are both tangent, in which case the curve of ingress will be tangent to the curve of egress on opposite sides of the surfaces, as in Fig. 161.

The nature of the curve of intersection may be determined in advance by drawing, under the same conditions as the auxiliary planes are drawn, tangent planes to the surfaces.

If in Fig. 159 we draw, as one of the auxiliary planes of this problem, the plane T tangent to the cylinder along the upper surface, the position of its vertical trace T–t' shows that such a plane intersects the cone, and that a portion of the cone remains entirely outside the cylinder.

The plane W drawn under the conditions mentioned above and tangent to the cylinder along the under surface does not cut the cone, showing that a portion of the cylinder remains outside the cone.

The curve of intersection then in this case will be one continuous curve, as shown in the figure.

If both the planes T and W had intersected the cone, the indication would be that the cylinder passed through the cone, giving two distinct curves of intersection.

If the base of the cone had fallen wholly within the two vertical traces T–t' and W–w', the indication would be that the cone passed through the cylinder, giving two distinct curves of intersection.

This last condition is illustrated in Fig. 160, where the cylinder whose axis is C–D intersects the cylinder whose axis is A–B in two distinct curves. The base of the cylinder whose axis is C–D falls wholly within the two planes T and U, which are drawn under the same conditions as the auxiliary planes of this problem, and tangent to the cylinder whose axis is A–B.

If the two vertical traces T–t' and W–w' in Fig. 159 had included and had been tangent to the base of the cone, the indication would be that the two surfaces were included by the tangent planes bringing the curves of intersection into a position of tangency. This condition is illustrated in Fig. 161, where the first cone whose axis is A–B intersects the second cone in two tangent curves. In Fig. 161 the two planes whose intersections with the plane of the base of the first cone are E–U and E–W are tangent to the first cone and are also tangent to the second cone, as may be seen from the revolved position of the lines in which these tangent planes intersect S, the plane of the base of the second cone.

409. Problem 278. *Find the intersection of a cylinder and a cone when the bases of both surfaces are on V.*

410. Problem 279. *Find the intersection of two cylinders when the base of one surface is on H and the base of the other is on V.*

411. Problem 280. *Find the intersection of two cones when the bases of both surfaces are on H.*

412. Problem 281. *Find the intersection of two cylinders of the same diameter, one horizontal and the other vertical, axes intersecting.*

413. Problem 282. *Find the intersection of two cylinders of unequal diameters, one horizontal and the other vertical, axes intersecting.*

414. Problem 283. *Find the intersection of two cylinders of unequal diameters, one horizontal and the other vertical, axes not intersecting.*

415. Problem 284. *Find the intersection of two cylinders of unequal diameters, the larger vertical, the smaller inclined at an angle of* 60 *degrees to H, axes intersecting.*

416. Problem 285. *To find the intersection of a sphere and a cylinder and to draw a rectilinear tangent to the curve of intersection at a given point on the curve.*

CASE 1. *To find the intersection.*

Analysis 1. Pass the auxiliary planes parallel to the axis of the cylinder, cutting from the cylinder elements and cutting from the sphere circles.

Analysis 2. If the base of the cylinder is a circle, we may pass the auxiliary planes parallel to this base, cutting circles from both surfaces.

Construction. In Fig. 162 let C represent the center of the sphere and let A–B, passing through the center of the sphere, represent the axis of the cylinder.

Pass the auxiliary planes parallel to A–B and perpendicular to H. The horizontal trace of the auxiliary plane containing the axis is S–$s_{\prime}$. This plane cuts the sphere in a great circle and cuts the cylinder in two elements which intersect the circumference of the circle in four points of the required curve of intersection.

To find these points, revolve the plane S about S–$s_{\prime}$ into H. C will fall at c_{II} and the great circle will fall at e_{II}–g_{II}–l_{II}–k_{II}. The axis A–B will fall at $a_{\prime}$–b_{II}, and the two elements in which the auxiliary plane intersects the cylinder will fall at $d_{\prime}$–e_{II} and $f_{\prime}$–g_{II},

parallel to $a_{\prime}$–b_{H}. The points E, K, G, and L, projected at ($e_{\prime}$, e'), ($k_{\prime}$, k'), ($g_{\prime}$, g'), and ($l_{\prime}$, l'), are the four points of intersection.

The auxiliary plane T cuts the sphere in a small circle whose center is M and whose diameter is equal to $n_{\prime}$–$o_{\prime}$. The same plane

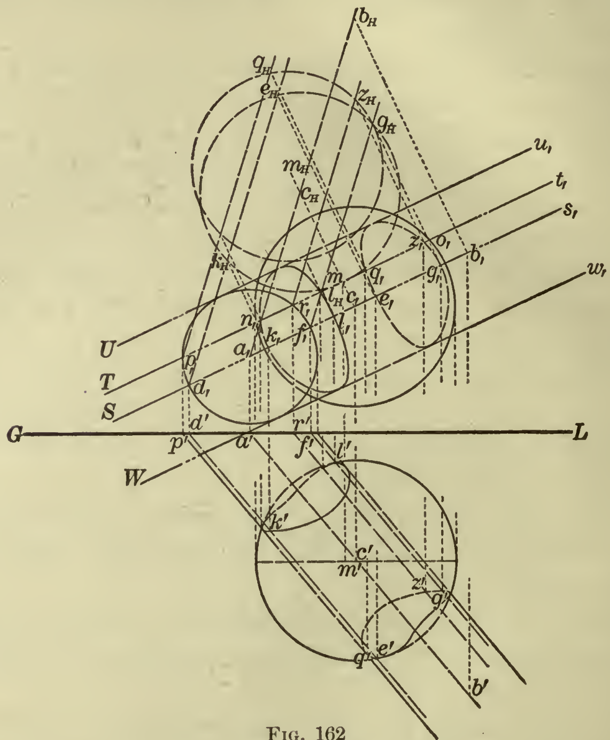

Fig. 162

T cuts the cylinder in two elements, P–Q and R–Z, which are parallel to A–B.

By passing a sufficient number of auxiliary planes we may locate enough points to determine the required curve of intersection.

Since the auxiliary planes U and W, which are drawn tangent to the cylinder, both intersect the sphere, there are, according to Section 408, two distinct curves of intersection, as may be seen from the drawing.

The visible and invisible portions of the curves may be determined by methods previously explained.

Case 2. *To draw a rectilinear tangent to the curve of intersection.*

The rectilinear tangent to the curve at any given point will be the intersection of two planes, one tangent to the cylinder at the given point and the other tangent to the sphere at the same point.

417. Problem 286. *Find the intersection of a sphere and a cylinder when the axis of the cylinder is vertical and passes through the center of the sphere.*

418. Problem 287. *Find the intersection of a sphere and a cylinder when the axis of the cylinder is horizontal and passes through the center of the sphere.*

419. Problem 288. *Find the intersection of a sphere and a cylinder when the axis of the cylinder does not pass through the center of the sphere, and when some of the elements of the cylinder remain wholly outside the sphere.*

420. Problem 289. *To find the intersection of a hemisphere and a cone and to draw a rectilinear tangent to the curve of intersection.*

Case 1. *To find the intersection.*

Analysis 1. Pass the auxiliary planes through the vertex of the cone and perpendicular to H, cutting elements from the cone and semicircles from the hemisphere.

Analysis 2. Same as Analysis 2 of Section 416.

Construction. Let the cone and the hemisphere be represented as in Fig. 163, where the vertex of the cone is taken at the center of the hemisphere.

The plane S represents one of the auxiliary planes, cutting the cone in two elements B–R and B–P, and cutting the hemisphere in a semicircle.

To find the points in which these elements intersect the semicircle, revolve S about a vertical axis through B until it is parallel to V. The two elements B–R and B–P will then be vertically projected at b'–r'' and b'–p'' respectively, and the vertical projection of the semicircle will be coincident with the vertical projection of the hemisphere. Therefore u'' and q'' are the vertical projections of the revolved positions of two points of the required curve of intersection.

The auxiliary plane T, parallel to V, cuts the cone in two elements B–K and B–L, and cuts the hemisphere in a semicircle

whose vertical projection coincides with the vertical projection of the hemisphere. This plane will locate the points of the curve whose vertical projections fall on the vertical projection of the hemisphere.

In the same way other planes may be passed and a sufficient number of points may be determined to locate the curve of intersection.

When projecting on *H*, if we regard the hemisphere as hollow, the whole curve will be visible.

When projecting on *V*, that portion of the curve which lies on the front portions of both surfaces and between extreme elements will be visible.

CASE 2. *To draw a rectilinear tangent to the curve of intersection.*

A rectilinear tangent to the curve of intersection at any point will be the intersection of two planes, one of which is tangent to the hemisphere at the given point and the other tangent to the cone at the same point.

421. Problem 290. *To develop an oblique cone.*

Analysis. The intersection of the surface of any cone by the surface of a sphere whose center is at the vertex of the cone is a curve whose points are all equidistant from the vertex of the cone, because the curve lies on the surface of the sphere as well as on the surface of the cone.

For this reason, in the development of the cone, this particular curve of intersection will roll out into the arc of a circle whose center is the vertex of the cone and whose radius is equal to the radius of the sphere.

If upon this arc, starting from any point, we lay off successively the actual arc distances between the points in which the elements of the cone cut the curve of intersection before development, and if we connect these points of division and the center of the arc by straight lines, these straight lines will represent the elements of the cone in development.

We may now use these elements to determine points in the development of other curves on the surface of the cone.

Construction. Let it be required to develop the oblique cone given in Fig. 163. The intersection of the surface of the cone by

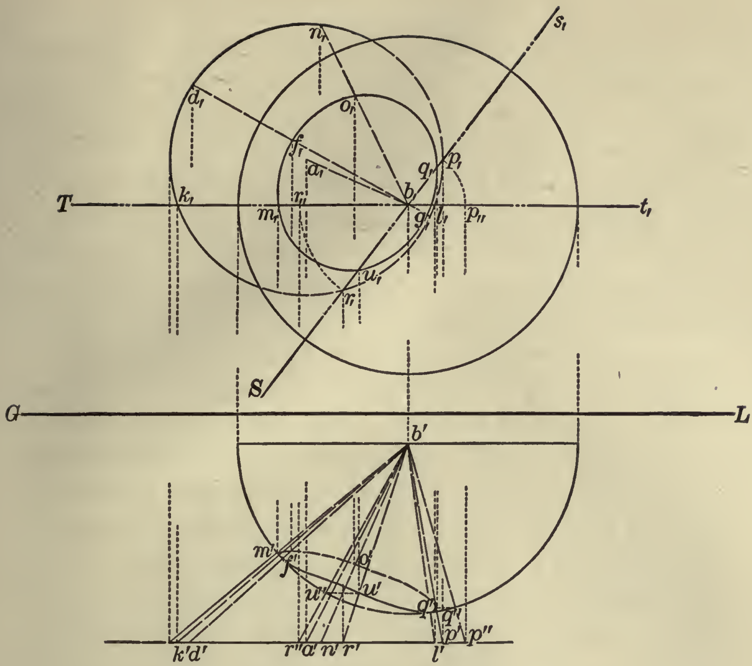

FIG. 163

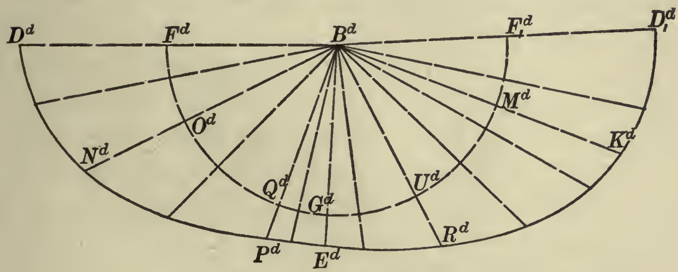

FIG. 164

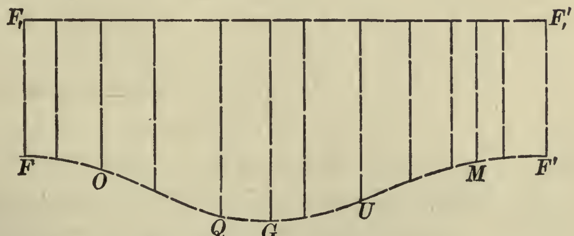

FIG. 165

the sphere is already determined. Take the plane which is tangent to the cone along the element B–D as the plane of development.

Then in Fig. 164 the arc F^d–O^d– $\cdots$ –F_i^d, with center at B^d and with radius equal to the radius of the sphere, may represent the indefinite development of the curve cut from the surface of the cone by the sphere.

To determine the positions in development of the points on this curve through which the various elements of the cone pass before development, we must determine the actual distances between these points measured along the curve before development.

To do this, develop the horizontal projecting cylinder of the original curve, as show in Fig. 163. The base of this cylinder is f_i–o_i–q_i– $\cdots$ –f_i, and since the plane of the base is perpendicular to the elements of the cylinder, the curve of the base will roll out into a straight line to which the elements of the cylinder will remain perpendicular (see Problem 249, Case 4).

The development of this cylinder is shown in Fig. 165, where F_i–F_i' represents the development of the curve of the base, and where F–O–Q– $\cdots$ –F' represents the development of the original curve of intersection. The actual distances between the points of division on this curve are now revealed.

Returning now to Fig. 164, start with F^d, any point on the arc, and make the distances F^d–O^d, O^d–Q^d, Q^d–G^d, etc., such that their rectified lengths shall be equal respectively to the rectified distances F–O, O–Q, Q–G, etc., of Fig. 165. Through B^d and the points just determined draw B^d–F^d–D^d, B^d–O^d–N^d, etc., to represent in development the elements of the cone passing through the points F, O, Q, etc., of Fig. 163.

Let it now be required to develop the curve of the base of the cone. In Fig. 164 make B^d–D^d, B^d–N^d, B^d–P^d, etc., equal respectively to B–D, B–N, B–P, etc., of the original cone, Fig. 163. The curve D^d–N^d–P^d–R^d–D_i^d represents the curve of the base of the cone in development. In the same way we may find the development of any curve on the surface of the cone.

422. Problem 291. *Find the intersection of a sphere by a cone when the axis of the cone contains the center of the sphere, and when the vertex of the cone is outside the surface of the sphere.*

423. Problem 292. *Find the intersection of a sphere and a cone when the vertex of the cone lies outside the surface of the sphere, and when some of the elements of the cone lie wholly outside the sphere.*

424. Problem 293. *Given a sphere and a point without the surface; required to determine a cone with vertex at the point and with surface tangent to the sphere, and to determine the circle of tangency between the two surfaces.*

425. Problem 294. *To find the intersection of any two surfaces of revolution when their axes intersect, and to draw a rectilinear tangent to the curve of intersection.*

CASE 1. *To find the intersection.*

Analysis. Draw a series of auxiliary spheres with centers at the point in which the axes intersect.

These spheres will cut from each surface the circumference of a circle whose plane will be perpendicular to the axis of the surface. The circumferences of these circles will intersect in points common to both surfaces, and therefore in their curve of intersection.

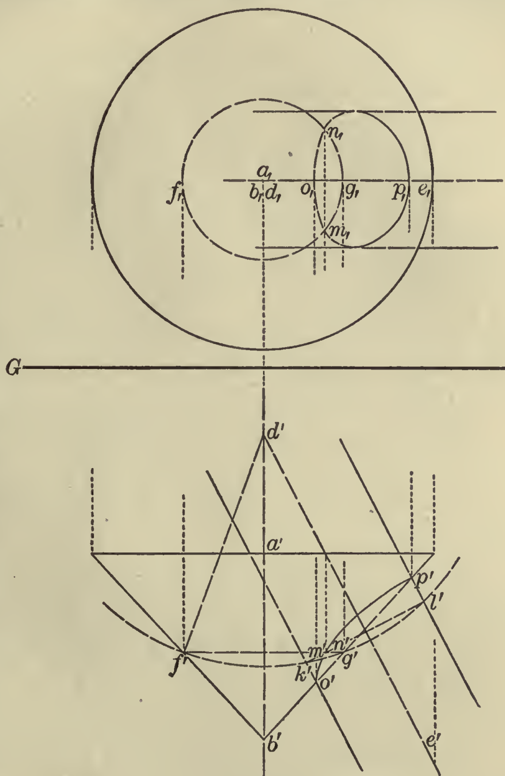

FIG. 166

Construction. Let it be required to find the intersection of an inverted right circular cone by a right circular cylinder, as represented in Fig. 166. *A–B* represents the axis of the cone and *D–E* represents the axis of the cylinder, the two axes intersecting at *D* and lying in a plane parallel to *V*.

An auxiliary sphere with center at D and with radius equal to $d'–f'$ will cut the surface of the cone in the circumference of a circle whose vertical projection is $f'–g'$ and whose horizontal projection is $f_{/}–m_{/}–g_{/}–n_{/}$. The same sphere will cut the surface of the cylinder in the circumference of another circle whose vertical projection is $k'–l'$. The circumferences of these two circles intersect in two points vertically projected at (m', n') and horizontally projected at $m_{/}$ and $n_{/}$ respectively. M and N are two points of the required curve of intersection, and others may be found in the same way.

The meridian plane of the axes will cut the cone and the cylinder in those elements which, when projecting upon V, appear as extreme elements. These elements intersect at O and P respectively, the lowest and the highest points of the curve of intersection.

When projecting on H, if we regard the inverted cone as hollow, that portion of the curve of intersection lying on the upper surface of the cylinder will be visible.

When projecting on V, that portion of the curve of intersection lying on the front surface of the cylinder and on the front surface of the cone will be visible.

CASE 2. *To draw a rectilinear tangent to the curve of intersection.*

The rectilinear tangent to the curve of intersection at any point will be the intersection of two planes, one tangent to the cone at the given point and the other tangent to the cylinder at the same point.

By use of methods already explained we may develop both of the given surfaces and show the character of the curve of intersection when rolled out into a plane surface.

426. Problem 295. *By the process just explained find the intersection of two cylinders of revolution whose axes intersect.*

427. Problem 296. *By the above process find the intersection of a cone of revolution and a sphere.*

428. Problem 297. *By the above process find the intersection of two cylinders of revolution, axes intersecting, the larger horizontal and the smaller vertical.*

CHAPTER XVIII

ISOMETRIC PROJECTION AND OTHER FORMS OF ONE-PLANE PROJECTION

429. Introductory Statements. In making a working drawing of a solid it is customary to place the object in such a position that the plane of two of its dimension lines is parallel to the plane of projection. This results in the representation of only one face of the object at a time.

It is sometimes desirable, however, to show three dimension faces of a solid in one view, and without going to the trouble of making a perspective drawing. This is accomplished by isometric projection.

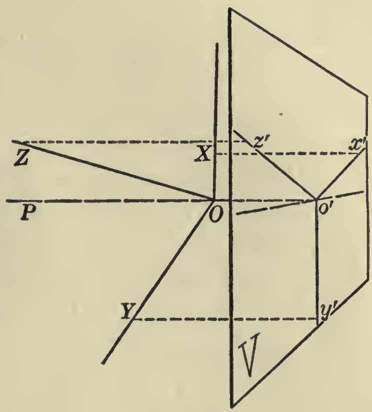

FIG. 167

In Fig. 167 let $O–X$, $O–Y$, and $O–Z$ represent the three edges of a right trihedral angle with vertex at O. Through the vertex O draw a straight line $P–O–o'$ equally inclined to the three edges. Through any point o' on $P–O–o'$ pass a plane V perpendicular to $P–O–o'$, and project the three edges of the right trihedral angle upon this plane. Since the edges themselves are equally inclined to the line $P–O–o'$, and therefore equally inclined to the plane V, and since they form the three edges of a right trihedral angle, the three projections $o'–x'$, $o'–y'$, and $o'–z'$ will radiate from o' so as to form angles of 120 degrees.

The lines $O–X$, $O–Y$, and $O–Z$ are called *coördinate axes*. The plane V is called the *isometric plane of projection;* the point o', the projection of O, is called the *isometric origin;* and the lines $o'–x'$, $o'–y'$, and $o'–z'$, which are the projections upon the plane V of the original axes $O–X$, $O–Y$, and $O–Z$, are called the *isometric axes.*

The planes determined by the lines $O-X$ and $O-Y$, $O-Y$ and $O-Z$, and $O-Z$ and $O-X$ are called *coördinate planes*, and the projections of magnitudes lying on these planes between the limiting axes will fall between the projections of these limiting axes.

The axis $O-Y$ is given a position directly beneath the line $P-O$. As a result the isometric axis $o'-y'$ has a vertical position, the axis $o'-x'$ makes an angle of 30 degrees with a horizontal line on the right, and the axis $o'-z'$ makes an angle of 30 degrees with a horizontal line on the left. For this reason the isometric axes and all straight lines parallel to them are easily represented upon the drawing board by use of the T-square and the 30-degree triangle.

Each of the three dimensions — length, breadth, and thickness — of a solid are measured on straight lines perpendicular to the plane of the other two, corresponding with the three edges of a right trihedral angle.

It is therefore possible to place a solid in such a position that its three principal dimension lines shall coincide with the three coördinate axes $O-X$, $O-Y$, and $O-Z$ of Fig. 167. When the magnitude occupies this position its projection upon the plane V will reveal the characteristics of three dimension faces of the object, and the projection is called *isometric*.

Whenever a solid is projected upon a plane to which its principal dimension lines are equally inclined, the projection is called isometric, since on account of this equality of inclination the projections of equal distances measured upon these dimension lines, or upon lines parallel to them, will be equal.

The projection of any magnitude may be said to be isometric whenever this projection is determined by reference to isometric axes which themselves are the isometric projections of the three coördinate axes to which the magnitude is referred, coördinates with reference to the isometric axes appearing as the isometric projections of the corresponding coördinates used in connection with the original coördinate axes.

430. The Isometric Scale. The inclination of each of the three coördinate axes $O-X$, $O-Y$, and $O-Z$ to the plane of projection is $35°.16'$. By trigonometry the isometric projection of one foot

measured upon either one of these axes is 1 foot times the natural cosine of 35°.16′, which is equal to .81647 ft.

If this length is divided into twelve equal parts, each one of these divisions will represent the isometric projection of an inch measured under the conditions mentioned above. In the same way we may subdivide the inch and thus establish an isometric scale by which we may determine the length in isometric projection of any distance measured upon the coördinate axes or upon any lines parallel to these axes.

Since drawings are usually made either on a smaller or a larger scale than the object represented, and since on account of the equal inclination of the coördinate axes to the plane of projection, the projections of distances measured along these lines are equally foreshortened, there is no good reason why any other than the ordinary foot scale should be used in connection with this form of projection. The isometric scale is therefore not used in practical drafting.

431. Shade Lines. In isometric projection the rays of light are assumed parallel to the plane of projection and inclined at an angle of 30 degrees with the horizon. With these exceptions, shade lines in isometric projection are determined and represented precisely as in orthographic projection.

If the cube to whose diagonal rays of light in orthographic projection have been referred (see Section 230) is so placed that the diagonal passing through the upper right-hand front vertex of the cube is perpendicular to V, and so that the right-hand vertical edge of the cube is still in a vertical plane, the diagonal passing through the upper left-hand front vertex of the cube in its first position — the diagonal to which rays of light in orthographic projection were referred — will in the new position be parallel to V and make an angle of 30 degrees with H.

In isometric projection, then, rays of light may be referred to the same diagonal of the cube as in orthographic projection, provided the cube occupies the position indicated above.

432. General Instructions in regard to Solution of Problems. The isometric projection of a point whose coördinates with reference to two coördinate axes are known is the intersection of the isometric projections of the two coördinate lines of the point.

The isometric projection of a point whose coördinates with reference to three coördinate planes are known is the intersection of the isometric projections of the three coördinate lines of the point.

The isometric projection of a straight line is the straight line determined by the isometric projections of two points of the given line.

Since the orthographic projections of parallel lines are parallel, the isometric projections of parallel lines will be parallel whether the lines themselves are parallel to the coördinate axes or not.

The isometric projection of a curved line is the line determined by the isometric projections of the points which determine the line.

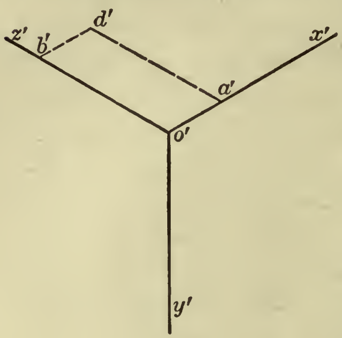

FIG. 168

The isometric projection of a surface determined or limited by lines, straight or curved, is that surface determined by the isometric projections of the determining or limiting lines.

To draw the isometric projection of a magnitude of three dimensions, place the magnitude in such a position that its three principal dimension lines shall be either coincident with or parallel to the three coördinate axes $O–X$, $O–Y$, and $O–Z$; then the isometric projections of the principal dimension lines of the magnitude will be either coincident with or parallel to the isometric axes.

433. Problem 298. *To find the isometric projection of a point whose coördinates with reference to two rectangular axes are known.*

In Fig. 168 let $o'–x'$, $o'–y'$, and $o'–z'$ represent the isometric axes.

Place the two rectangular axes to which the point is referred in coincidence with $O–X$ and $O–Z$ respectively. Then $o'–x'$ and $o'–z'$ are the isometric projections of the rectangular axes.

Make $o'–a'$ equal to the distance of the point from the axis $O–Z$, and through a' draw $a'–d'$ parallel to $o'–z'$.

Make $o'–b'$ equal to the distance of the point from the axis $O–X$, and through b' draw $b'–d'$ parallel to $o'–x'$.

The lines $a'–d'$ and $b'–d'$ are the isometric projections of the coördinate lines of the point, and the point d' is the isometric projection sought.

Of course the same result will be obtained if after drawing the line a'–d' we lay off upon it the remaining coördinate a'–d' equal to o'–b'.

The axes to which the point is referred may be assumed in coincidence with O–X and O–Y, or in coincidence with O–Z and O–Y.

434. Problem 299. *To find the isometric projection of a point whose coördinates with reference to three rectangular coördinate planes are known.*

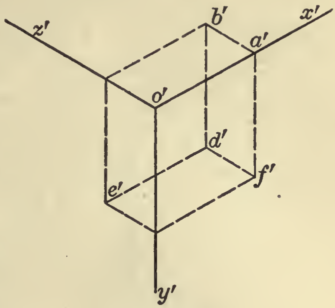

FIG. 169

In Fig. 169 let o'–x', o'–y', and o'–z' represent the isometric axes.

Place the three coördinate axes, to whose planes the point is referred, in coincidence with O–X, O–Y, and O–Z respectively. Then o'–x', o'–y', and o'–z' are the isometric projections of the coördinate axes.

Make o'–a' equal to the distance of the point from the plane Z–O–Y and through a' draw a'–b' parallel to o'–z'. Make a'–b' equal to the distance of the point from the plane X–O–Y, and through b' draw b'–d' parallel to o'–y'. Make b'–d' equal to the distance of the point from the plane Z–O–X. The point d' is the isometric projection of the point in question.

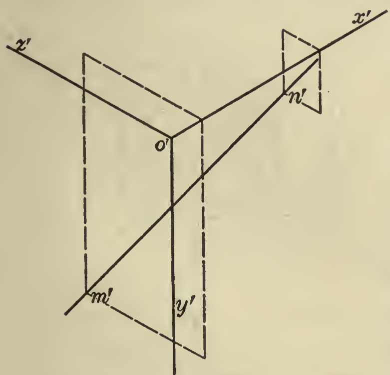

FIG. 170

The lines e'–d', d'–f', and b'–d' are the isometric projections of the three coördinate lines of the point D.

435. Problem 300. *To find the isometric projection of a straight line when the coördinates of two of its points with reference to three rectangular coördinate planes are known.*

In Fig. 170 let o'–x', o'–y', and o'–z' represent the isometric axes.

Place the coördinate axes to whose planes the line is referred in coincidence with O–X, O–Y, and O–Z. Then o'–x', o'–y', and o'–z' are the isometric projections of these coördinate axes.

By Section 434 find m' and n', the isometric projections of the two given points. The line m'–n' is the required projection.

436. Problem 301. *To find the isometric projection of a square card when the coördinates of its four vertices with reference to two rectangular axes are known.*

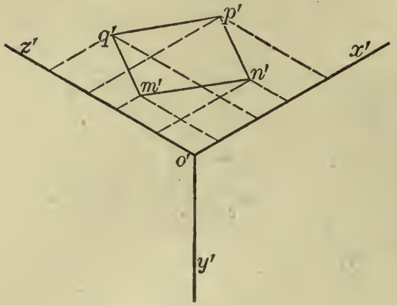

FIG. 171

In Fig. 171 let o'–x' and o'–z' represent the isometric projections of the two axes to which the card is referred.

By Section 433 find the isometric projections of the four vertices M, N, P, and Q.

The polygon m'–n'–p'–q' is the projection sought.

The projection may be made with reference to the axes o'–x' and o'–y', or with reference to the axes o'–y' and o'–z'.

437. Problem 302. *To find the isometric projection of a triangular card when the coördinates of its three vertices with reference to three rectangular coördinate planes are known.*

In Fig. 172 let o'–x', o'–y', and o'–z' represent the isometric projections of the axes to whose planes the card is referred.

By Section 434 find the isometric projections of the three vertices M, N, and P. The triangle m'–n'–p' is the projection sought.

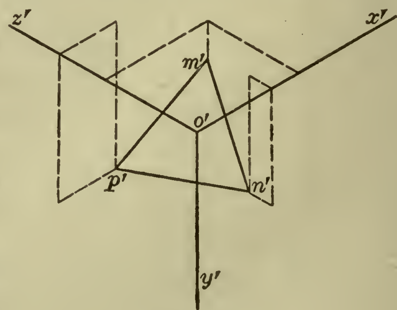

FIG. 172

438. Problem 303. *To make an isometric projection of a cube.*

See Fig. 173. Let o'–x', o'–y', and o'–z' represent the isometric axes. Place the cube so that one of its vertices shall coincide with the origin of coördinates, O, and so that its three adjacent edges shall coincide with the three coördinate axes O–X, O–Y, and O–Z.

The isometric projections of these three edges will fall on the isometric axes, and the isometric projections of the remaining edges will be parallel to these three axes.

Make $o'–a'$ and $o'–d'$ each equal to the edge of the cube. Through a' draw $a'–e'$ parallel to $o'–d'$. Through d' draw $d'–e'$ parallel to $o'–a'$ and intersecting $a'–e'$ at e'. The figure $o'–a'–e'–d'$ represents the isometric projection of the upper face of the cube.

Make $o'–b'$ equal to the edge of the cube. Through a' draw $a'–f'$ parallel to $o'–b'$. Through b' draw $b'–f'$ parallel to $o'–a'$ and intersecting $a'–f'$ at f'. The figure $o'–a'–f'–b'$ represents the isometric projection of the right-hand face of the cube.

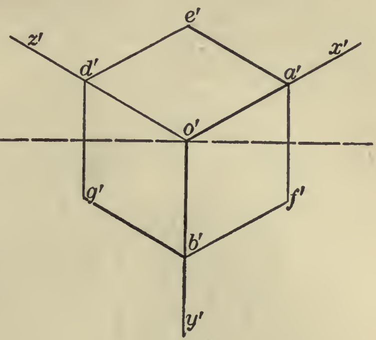

FIG. 173

Through b' draw $b'–g'$ parallel to $o'–d'$. Through d' draw $d'–g'$ parallel to $o'–b'$ and intersecting $b'–g'$ at g'. The figure $o'–b'–g'–d'$ represents the isometric projection of the left-hand face of the cube.

The figure $e'–a'–f'–b'–g'–d'$ is the isometric projection of the cube.

To determine the shade lines of the cube.

With the source of light in its new position, as stated above, the three faces $O–E$, $O–G$, and $G–E$ will be in the light, and the three remaining faces will be in the dark. Therefore the visible shade lines are $G–B$, $B–O$, $O–A$, and $A–E$.

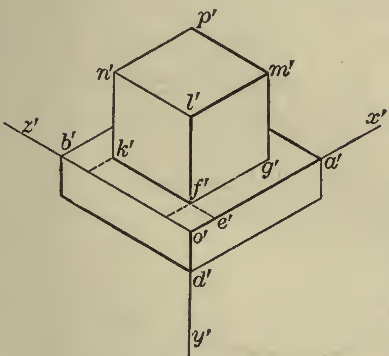

FIG. 174

439. Problem 304. *To make an isometric projection of a square pedestal supporting a square prismatic pillar.*

See Fig. 174. Place the upper front vertex of the pedestal at the origin O. Make $o'–a'$ and $o'–b'$ each equal to the side of the square pedestal; also make $o'–d'$ equal to the altitude of the pedestal. The pedestal is now completed by drawing through a', b', and d' straight lines parallel to the proper axes.

Knowing the relation of the horizontal dimensions of the pillar to the horizontal dimensions of the pedestal, we can determine the distance between the corresponding faces of the pillar and pedestal. Make $o'–e'$ equal to the distance between the left-hand face of the pillar and the left-hand face of the pedestal, and through e' draw $e'–f'–k'$ parallel to $o'–z'$. Make $e'–f'$ equal to the distance between the right-hand face of the pillar and the right-hand face of the pedestal, and through f' draw $f'–g'$ parallel to $o'–x'$. Make $f'–k'$ and $f'–g'$ each equal to the side of the square pillar. Through f', k', and g' draw the vertical straight lines $f'–l'$, $k'–n'$, and $g'–m'$. Make $f'–l'$ equal to the altitude of the pillar and complete the drawing.

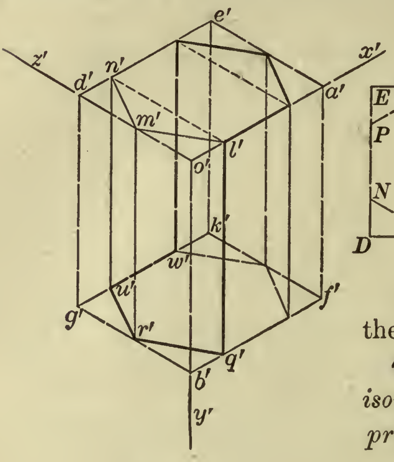

Fig. 175

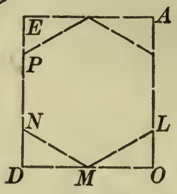

Fig. 176

440. Problem 305. *To make an isometric projection of an hexagonal prism.*

Analysis. Imagine the hexagonal prism to be inscribed within a rectangular prism whose rectangular bases circumscribe the hexagonal bases of the given prism, and whose altitude is equal to that of the given prism.

Draw the isometric projection of the rectangular prism and then by reference to this prism make the isometric projection of the hexagonal prism.

Construction. See Figs. 175 and 176. Let $L–M–N–P– \cdots –L$, Fig. 176, represent the hexagonal base of the given prism, and let $O–D–E–A$, of the same figure, represent the circumscribing rectangle of this base.

The isometric projection of the circumscribing prism is represented in Fig. 175 by $o'–a'–e'–d'–b'$, where $o'–a'$ and $o'–d'$ are equal respectively to the horizontal dimensions of this prism, and where $o'–b'$ is equal to the altitude of the prism.

To draw the isometric projection of the upper base of the hexagonal prism, make o'–l', o'–m', d'–n', etc., of Fig. 175 equal respectively to O–L, O–M, D–N, etc., of Fig. 176, and connect the points thus found by straight lines.

To complete the projection of the prism, find the isometric projection of the lower base by the process just explained and connect the corresponding vertices of the two bases.

The shade lines.

The upper base and the three faces L–R, M–U, and N–W are in the light, locating the shade lines as shown in the drawing.

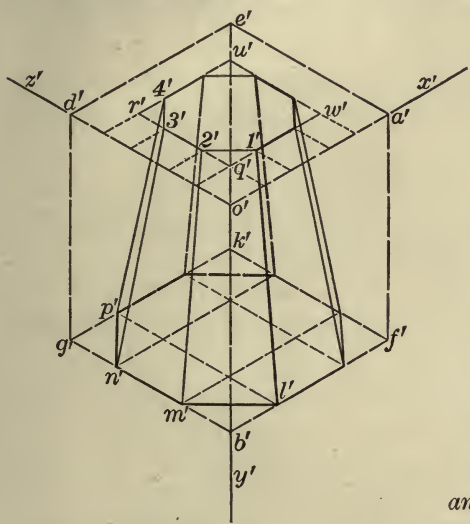

FIG. 177

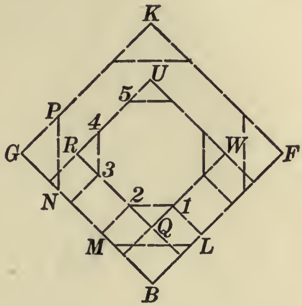

FIG. 178

441. Problem 306. *To make an isometric projection of the frustum of an octagonal pyramid.*

Analysis. Imagine the frustum of the pyramid to be inscribed within a square prism whose base is the circumscribing square of the larger base of the frustum and whose altitude is the altitude of the frustum.

Construction. See Figs. 177 and 178. Fig. 178 represents in plan the two bases of the frustum and their circumscribing squares. B–G–K–F represents the circumscribing square of the larger base and L–M–N–P– $\cdots$ represents the larger base itself.

Q–R–U–W represents the circumscribing square of the smaller base and 1–2–3–4– $\cdots$ represents the smaller base itself.

The isometric projection of the circumscribing prism is represented in Fig. 177 by o'–a'–e'–d'–b', where o'–a' and o'–d' are each made equal to the side of the circumscribing square of the larger base of the frustum, and where o'–b' is made equal to the altitude of the frustum.

The isometric projection of the lower base of the frustum is represented in Fig. 177 by l'–m'–n'–p'–$\cdots$, where b'–l', b'–m', b'–n', etc., are made equal respectively to B–L, B–M, B–N, etc., of Fig. 178.

The isometric projection of the circumscribing square of the upper base of the frustum is represented in Fig. 177 by q'–r'–u'–w', concentric with o'–d'–e'–a' and having its side equal to the side of the square Q–R–U–W of Fig. 178.

The isometric projection of the upper base of the frustum is represented in Fig. 177 by $1'$–$2'$–$3'$–$4'$–$\cdots$, determined as previously explained.

The isometric projection of the edges of the frustum may now be represented by connecting the vertices of the upper base with the corresponding vertices of the lower base.

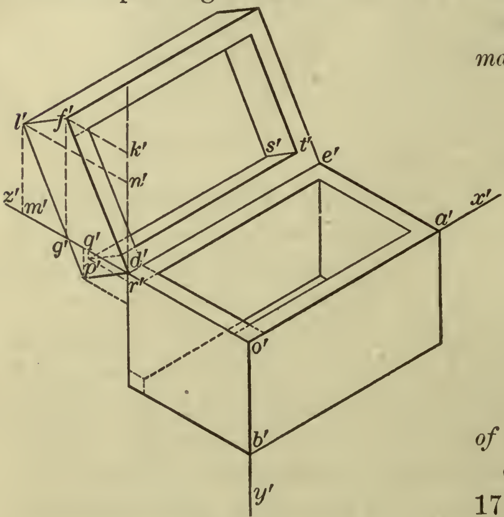

Fig. 179

442. Problem 307. *To make an isometric drawing of a box with open lid.*

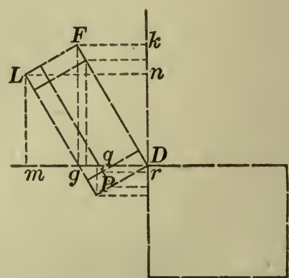

Fig. 180

Construction. See Figs. 179 and 180. The isometric projection of the box is represented in Fig. 179 by o'–a'–e'–d'–b'.

Fig. 180 represents an end view of the box and lid. By reference to a horizontal and a vertical axis through D in this figure we can

determine the coördinates of any point on the lid with reference to a horizontal and to a vertical plane through the hinge axis $D–E$.

The point F, for example, one apex of the lid, is at the distance $F–g$ above the horizontal plane and at the distance $F–k$ back of the vertical plane. The point F is also in the plane of the left-hand end of the box, and in isometric projection will be located with reference to the isometric axes $o'–z'$ and $o'–y'$.

In Fig. 179 make $d'–g'$ equal to $D–g$ of Fig. 180, and through g' draw $g'–f'$ parallel to the axis $o'–y'$. Lay off upon $g'–f'$ from g' the distance $g–F$ of Fig. 180. The point f' is the isometric projection of F.

In this way the isometric projection of any point on the lid may be determined.

The isometric projection of S, one of the inner vertices of the lid, is found thus: In the end view, Fig. 180, the point S will appear at P, at the distance $P–q$ below the horizontal plane and at the distance $P–r$ back of the vertical plane.

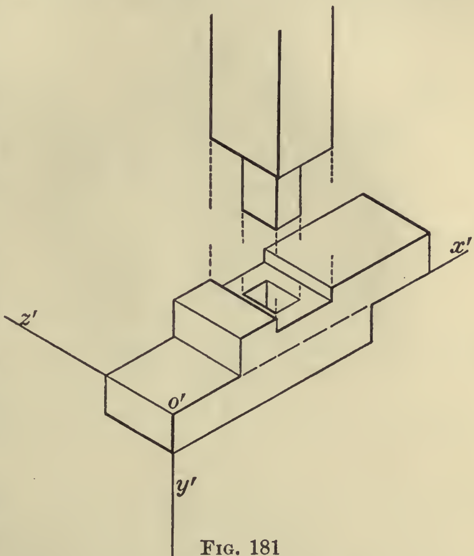

FIG. 181

In Fig. 179 make $d'–q'$ and $q'–p'$ equal respectively to $D–q$ and $q–P$ of Fig. 180, locating p'. Through p', which is the isometric projection of P, draw $p'–s'$ parallel to $o'–x'$, to represent the inner horizontal edge of the lid.

Through t' previously located draw $t'–s'$ parallel to $f'–l'$, to represent the inner vertical edge of the lid.

The point s', the intersection of $p'–s'$ and $t'–s'$, is the isometric projection of S.

443. Problem 308. *To represent a mortise and tenon. See Fig. 181.*

444. The Isometric Projection of Curved Lines. The isometric projection of curved lines may be determined by finding the

isometric projections of a number of the points of the curve and by tracing the required curve through these projections.

If the curve is of single curvature, locate, by reference to two rectangular axes, a sufficient number of points on the curve to determine the curve accurately. Find the isometric projections of the rectangular axes and then by use of the known coördinate distances determine the isometric projections of the points which locate the curve.

If the curve is of double curvature, locate, by reference to three rectangular coördinate planes, a sufficient number of points on the

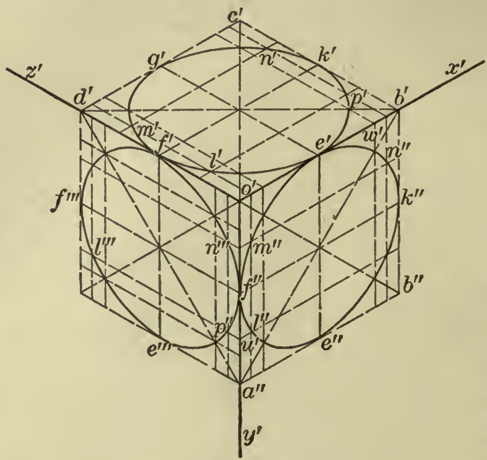

Fig. 182

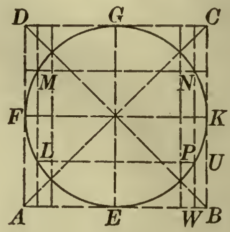

Fig. 183

curve to determine it.

Find the isometric projections of the coördinate axes and then by use of the known coördinates, referred to these axes, determine the isometric projections of the points which locate the curve.

445. Problem 309. *To make an isometric drawing of a circle.*

See Figs. 182 and 183. Fig. 183 represents the circle referred to the four sides of a circumscribing square A–B–C–D, any two of whose adjacent sides may be taken as rectangular axes of reference. The coördinates of a number of points, E, L, F, M, G, N, K, P, on the circumference, including the points of tangency, are here determined.

Fig. 182 shows the isometric projection of the circle in three positions. First, when the isometric projections of the axes A–B and A–D coincide with the isometric axes o'–x' and o'–z' respectively. Second, when the isometric projections of the axes D–C and D–A

coincide with the isometric axes o'–x' and o'–y' respectively. Third, when the isometric projections of the axes C–D and C–B coincide with the isometric axes o'–z' and o'–y' respectively.

In each case the isometric projections of the points are found by use of the coördinate measurements made in Fig. 183. For example, the point p', Fig. 182, is obtained by making o'–w' and w'–p' equal respectively to A–W and W–P of Fig. 183.

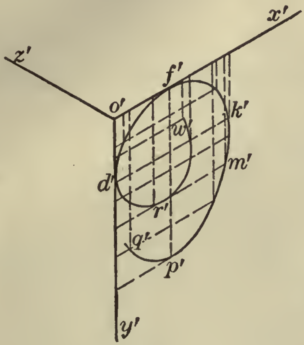

FIG. 184

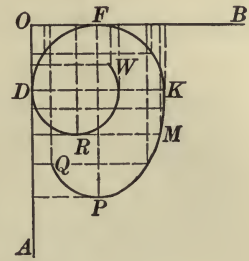

FIG. 185

The point l''' is located by making o'–u' and u'–l''' equal respectively to C–U and U–L of Fig. 183.

446. Problem 310. *To make an isometric drawing of any plane curve.*

See Figs. 184 and 185. Let Fig. 185 represent the curve when referred to two rectangular axes, one of which is O–A tangent to the curve at D and the other O–B tangent to the curve at F.

The coördinates of a number of points, W, R, D, F, M, Q, on the curve are determined with reference to these axes.

The isometric projection of this curve is shown in Fig. 184, where the isometric projections of the axes O–A and O–B coincide with the isometric axes o'–y' and o'–x' respectively, and where the isometric projections of the points W, R, D, etc., are found as previously explained.

We may draw the isometric projection of this curve when the curve occupies any desired position with reference to the coördinate axes O–X, O–Y, and O–Z, whether in the plane determined by any two of these axes or in the space-angle determined by the three axes.

447. Problem 311. *To make an isometric drawing of any curve of double curvature when referred to three rectangular coördinate planes.*

Let it be required to make an isometric drawing of the helix, when the helix is referred to the three orthographic planes of projection P, V, and H.

Fig. 187 represents the orthographic projection of the helix, from which the three coördinate lines of any point of the curve may be determined.

The isometric projection of the curve is shown in Fig. 186, where x'–o'–y' represents the isometric projection of the plane V, where

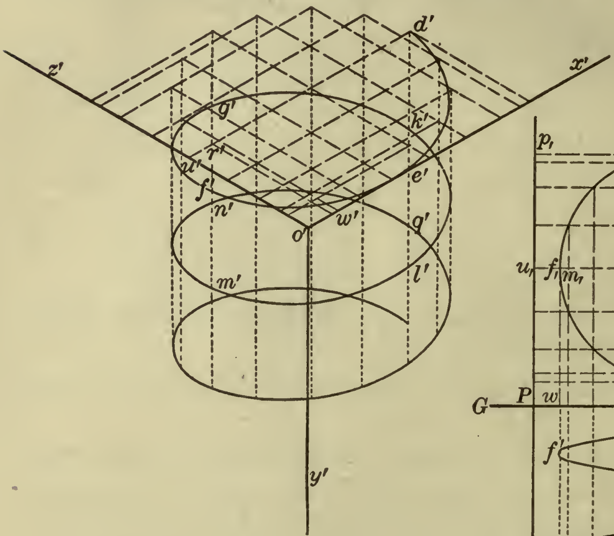

Fig. 186

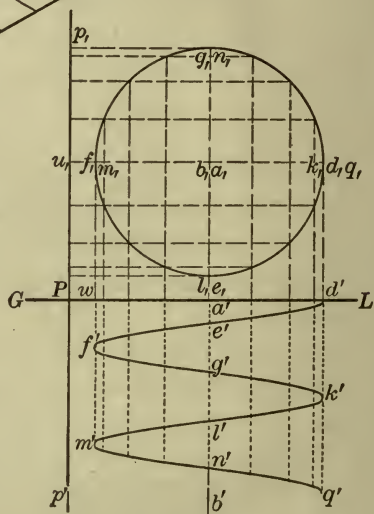

Fig. 187

x'–o'–z' represents the isometric projection of the plane H, and where z'–o'–y' represents the isometric projection of the plane P.

The isometric axis o'–x' represents the isometric projection of G–L, o'–z' represents the isometric projection of P–$p_{,}$, and o'–y' represents the isometric projection of P–p'.

The isometric projection of any point of the curve, for example the point M, may be found thus:

Fig. 187 shows that the distance of M from the plane P is $m_{\prime}-u_{\prime}$, that its distance from V is $m_{\prime}-w$, and that its distance from H is $w-m'$.

In Fig. 186 make $o'-w'$ equal to the distance of M from the plane P, and through w' draw $w'-r'$ parallel to $o'-z'$. Make $w'-r'$ equal to the distance of M from V, and through r' draw $r'-m'$ parallel to $o'-y'$. Make $r'-m'$ equal to the distance of M from H. The point m' is the isometric projection of M, and other points may be found in the same way.

448. Oblique Projection. In orthographic and in isometric projection the observer is assumed at an infinite distance from the plane of projection, and the resultant parallel projecting lines are taken perpendicular to the plane of projection.

If the projecting lines are taken oblique to the plane of projection, the system of projection is called *oblique projection.*

449. Cavalier Perspective or Cabinet Projection. When the projecting lines are assumed parallel to each other and at an inclination of 45 degrees to the plane of projection, the system of projection is called *cavalier perspective* or *cabinet projection.*

In Fig. 188 let $a'-B$ represent a straight line perpendicular to V and piercing it at a'. Through B draw a projecting line $B-b'$ at an inclination of 45 degrees to V and piercing V at b'. The line $a'-b'$ is the cabinet projection of $a'-B$.

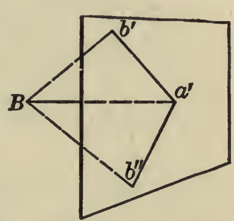

Fig. 188

No restriction whatever is made upon the direction which the projecting lines shall take so long as they are inclined at an angle of 45 degrees with the plane of projection. In any given problem, however, the projecting lines must be parallel.

450. Observations. Since the projecting lines are inclined 45 degrees to the plane of projection, the cabinet projection of a straight line perpendicular to the plane of projection will be equal to the line itself.

The cabinet projection of a straight line in the plane of projection will be the line itself.

The cabinet projection of a straight line parallel to the plane of projection will be equal to the line itself.